Multiple integrals

Recall that the integral $\int_a^b f(x)dx$ is the signed area under the graph (Fig 1)

Fig 1

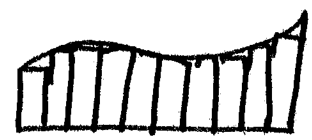
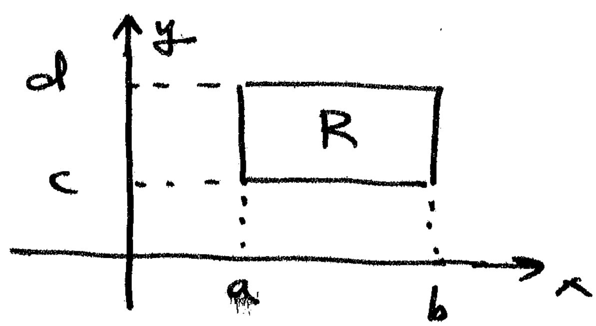


Fig 2

and that the integral can be approximated by Riemann sums (Fig. 2).

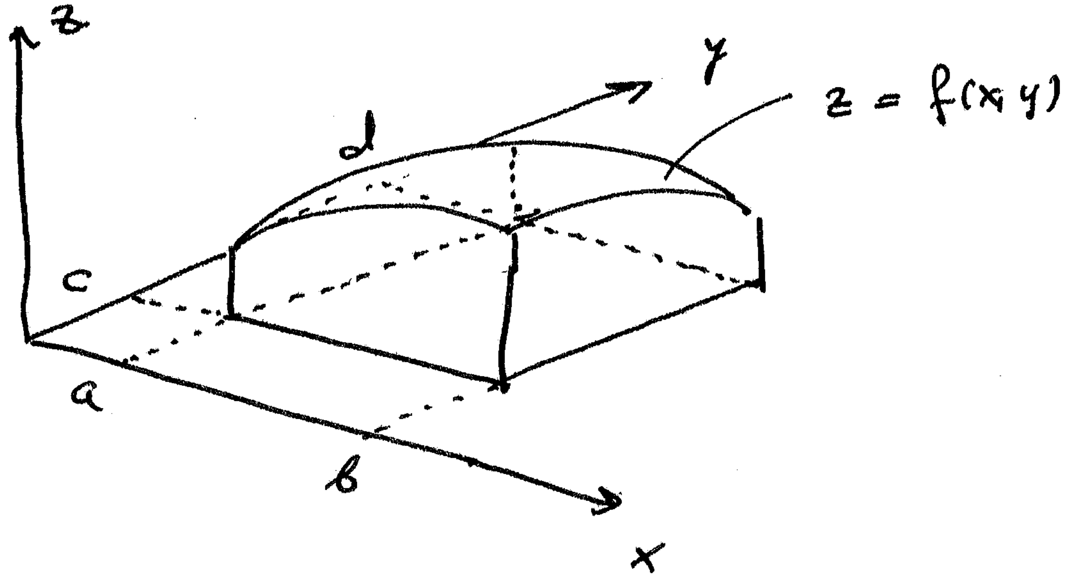
Similarly the double integral of a continuous function $z = f(x,y)$ over a rectangular region

$$R = [a,b] \times [c,d] = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} a \leq x \leq b \\ c \leq y \leq d \end{matrix} \right\}$$



is the volume of the solid region between the graph and the xy plane.

$$S = \{ (x,y,z) \mid (x,y) \in R,\ 0 \leq z \leq f(x,y) \}$$



If part of the graph is under the xy-plane i.e. $f(x,y)<0$, then the corresponding volume is taken with the negative sign. The volume can be approximated by Riemann sums in analogy to the case of functions of 1-variable (see the textbook)

The double integral of f over R is denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

Theorem (Fubini)

If $f(x,y)$ is a continuous function defined on a rectangle

$$R = \{(x,y) \mid a \leq x \leq b, \ c \leq y \leq d\}$$

then

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

When we compute the integral $\int_c^d f(x,y)\,dy$ we regard x as a fixed parameter so we actually integrate a function of one variable y, i.e. we integrate the function $g_x(y) = f(x,y)$. The subscript x in g_x indicates that the function g of variable y depends on the parameter x. Hence the resulting integral

$$\int_c^d f(x,y)\,dy = \int_c^d g_x(y)\,dy := h(x)$$

will also depend on the value of the parameter x i.e. it will define a function of x. Let us denote it by $h(x)$.

Hence

$$\int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{h(x)} \Big)\,dx = \int_a^b h(x)\,dx.$$

Thus computing a double integral reduces to computation of two integrals of one variable:

1. First: $\displaystyle\int_c^d f(x,y)\,dy := h(x)$

   Then $\displaystyle\int_a^b h(x)\,dx.$

Similar procedure applies to the integration

$$\int_c^d \Big(\int_a^b f(x,y)\,dx \Big)\,dy$$

2. First $\displaystyle\int_a^b f(x,y)\,dx := u(y)$

   Then $\displaystyle\int_c^d u(y)\,dy.$

The Fubini theorem says that it does not matter in what order we integrate. Both methods 1 and 2 will give us the same answer which is equal to

$$\iint_R f(x,y)\,dA.$$

Example Evaluate

$$\iint_R (x-3y^2)\,dA, \qquad R=\{(x,y)\,|\,0\le x\le 2,\ 1\le y\le 2\}$$

Solution We will compute the integral using both methods 1 and 2 and we will see that we will get the same answer.

1 $$\iint_R (x-3y^2)\,dA = \int_0^2\left(\int_1^2 (x-3y^2)\,dy\right)dx$$

$$= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx =$$

$$= \int_0^2 \left[(2x-8)-(x-1)\right]dx = \int_0^2 x-7\,dx$$

$$= \frac{x^2}{2} - 7x\Big|_0^2 = -12$$

(2) $\iint_R (x-3y^2)\,dA = \int_1^2 \left(\int_0^2 (x-3y^2)\,dx \right) dy$ (6)

$= \int_1^2 \left[\frac{x^2}{2} - 3y^2 x \right]_{x=0}^{x=2} dy =$

$= \int_1^2 \left[\left(\frac{2^2}{2} - 3y^2 \cdot 2 \right) - 0 \right] dy =$

$= \int_1^2 2 - 6y^2\,dy = 2y - 2y^3 \Big|_1^2 = -12.$

Example Evaluate

$\iint_R y \sin(xy)\,dA$, $R = [1,2] \times [0, \pi]$
$= \{(x,y) \mid 1 \le x \le 2, 0 \le y \le \pi\}$

Solution

Using method (1) leads to

$\iint_R y \sin(xy)\,dA = \int_1^2 \Big(\underbrace{\int_0^\pi y \sin(xy)\,dy}_{\text{not easy to compute this integral}} \Big) dx$

However, method ② leads to an easy computation ⑦

$$\iint_R y \sin(xy)\, dA = \int_0^{\pi} \left(\int_1^2 y \sin(xy)\, dx \right) dy$$

$$= \int_0^{\pi} \Big[-\cos(xy) \Big]_{x=1}^{x=2} dy =$$

$$= \int_0^{\pi} \big[-\cos(2y) + \cos y \big]\, dy =$$

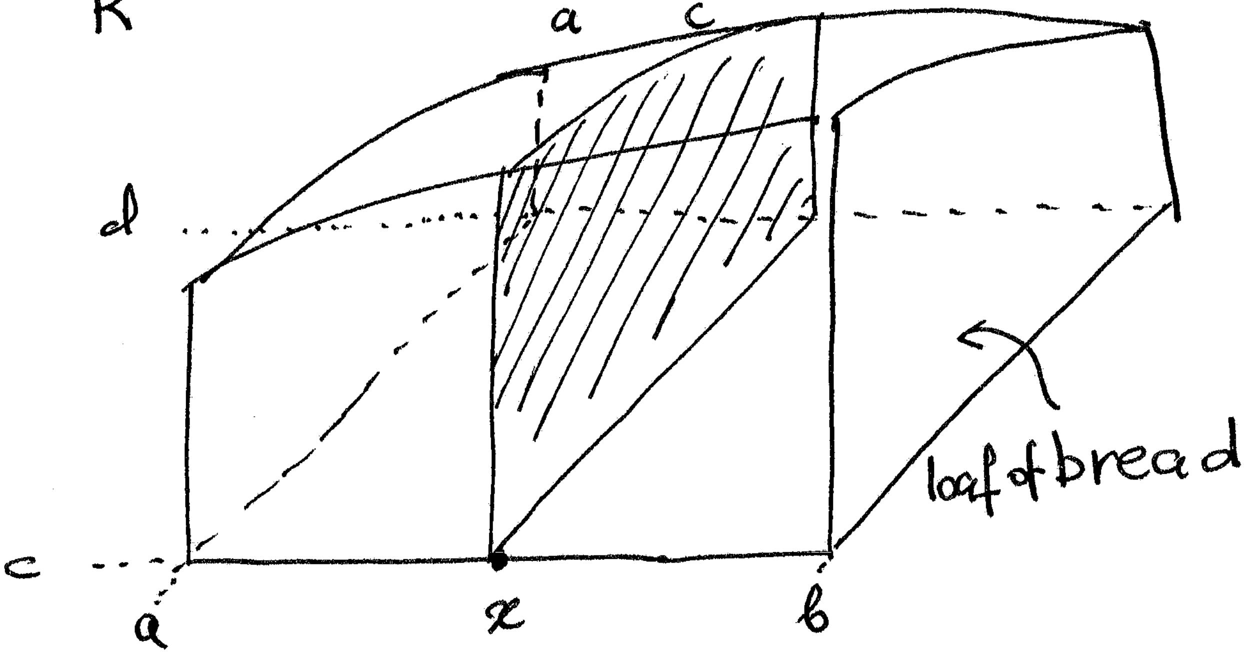
$$= -\frac{1}{2}\sin(2y) + \sin y \Big|_0^{\pi} = 0.$$

<u>Remark</u> The Fubini theorem allows you to compute the integral in two different ways. In many situations both methods are easy, but in some other cases one method is easy while the other one is nearly impossible.

Geometric interpretation

Recall that

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$



$$h(x) = \int_c^d f(x,y)\,dy = \text{the shaded area}$$

Thus

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_a^b \underbrace{h(x)}_{\text{the shaded area}}\,dx$$

Imagine that the solid under the graph of $z = f(x,y)$ is a loaf of bread.

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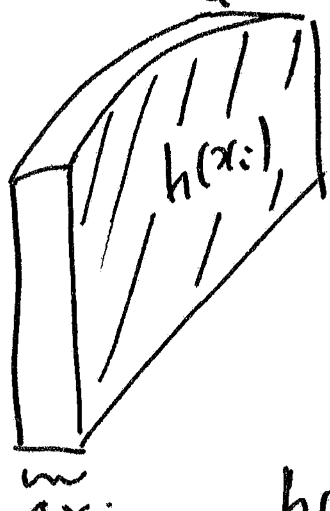
The double integral represents the volume of this loaf of bread.

Riemann sum approximation

$$\iint_R f(x,y)\,dA = \int_a^b h(x)\,dx \approx \sum_{i=1}^{n} h(x_i)\,\Delta x_i$$



$h(x_i)$ – the area of the slice of bread

Δx_i – thickness of the slice

$h(x_i)\Delta x_i \approx$ volume of the slice of bread

Thus

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{n} h(x_i)\,\Delta x_i$$

volume of the loaf of bread

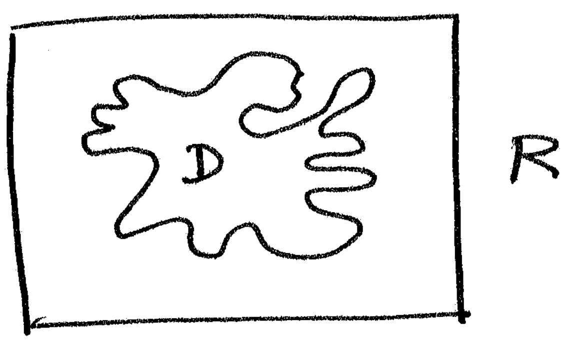
sum of volumes of slices of bread.

Integration over general regions

We know how to compute the integral $\iint_R f(x,y)\,dA$ when $R=[a,b]\times[c,d]$ is a rectangle.

The question is how to integrate over regions that are not rectangular. For example we need to know how to integrate a function that is defined in a disc.

The method is as follows. Given a continuous function $f(x,y)$ defined on a domain D, we find a larger rectangle R that contains D

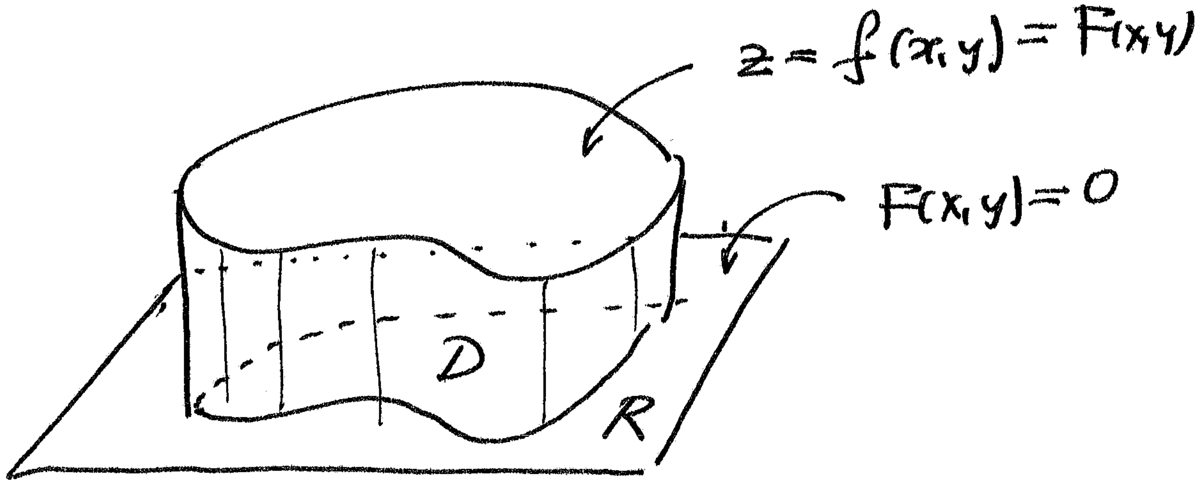


and we define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

and

$$\iint_D f(x,y)\,dA \overset{\text{definition}}{=\!=} \iint_R F(x,y)\,dA.$$



This is the graph of $z = F(x,y)$, because outside D the function $F(x,y)$ equals 0.

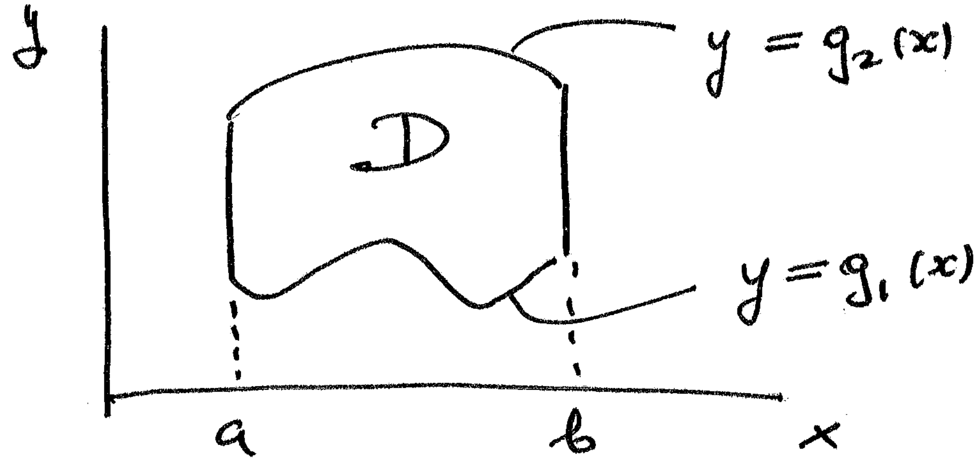
Thus the volume under the graph of F equals the volume under the graph of f because the part which is outside of D does not contribute to the volume – the function equals 0 in that part. Hence

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \text{volume under } F$$
$$= \text{volume under } f.$$

Now we will learn how to compute such integrals.

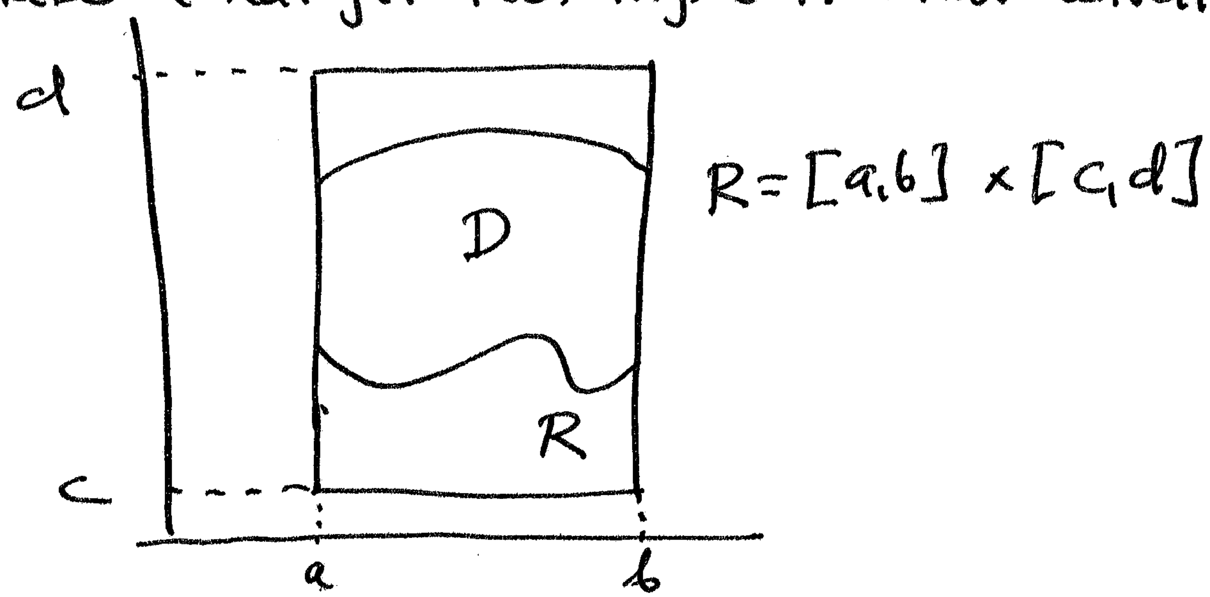
Domains of type I

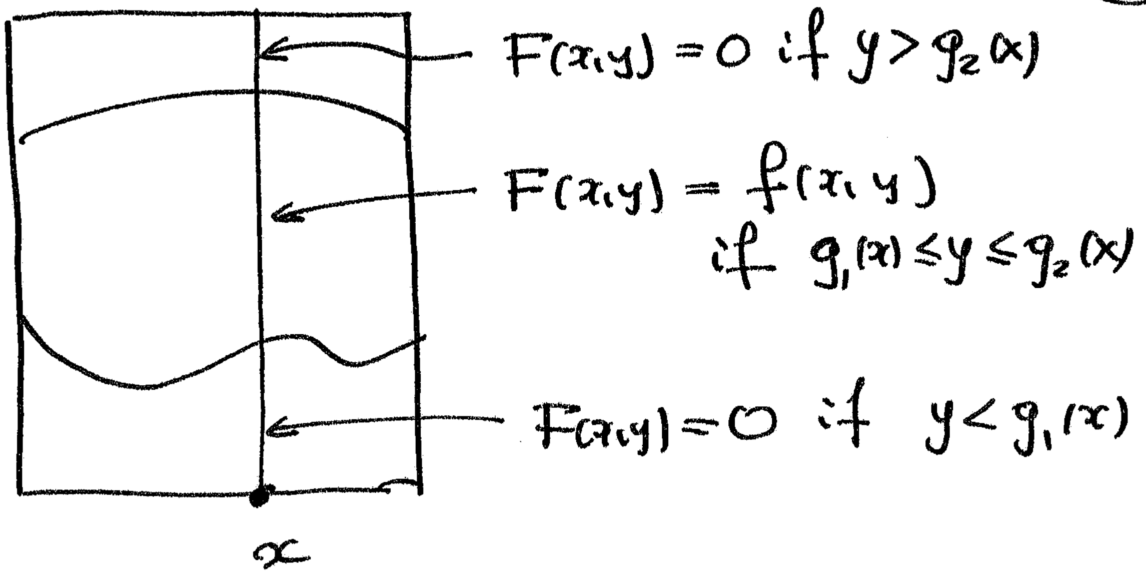
Suppose that a domain D is between two graphs $y = g_1(x)$ and $y = g_2(x)$



$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

Take a larger rectangle R that contains D





$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA =$$

$$\int_a^b \left(\int_c^d \underbrace{F(x,y)\,dy}_{\substack{0 \text{ if } y>g_2(x) \\ 0 \text{ if } y<g_1(x)}} \right) dx =$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \underbrace{F(x,y)}_{\substack{f(x,y) \text{ if} \\ g_1(x)\le y\le g_2(x)}}\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

We have :

<u>Theorem</u>

If $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx .$$

Domains of type $\underline{\text{II}}$ are defined in a similar way

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

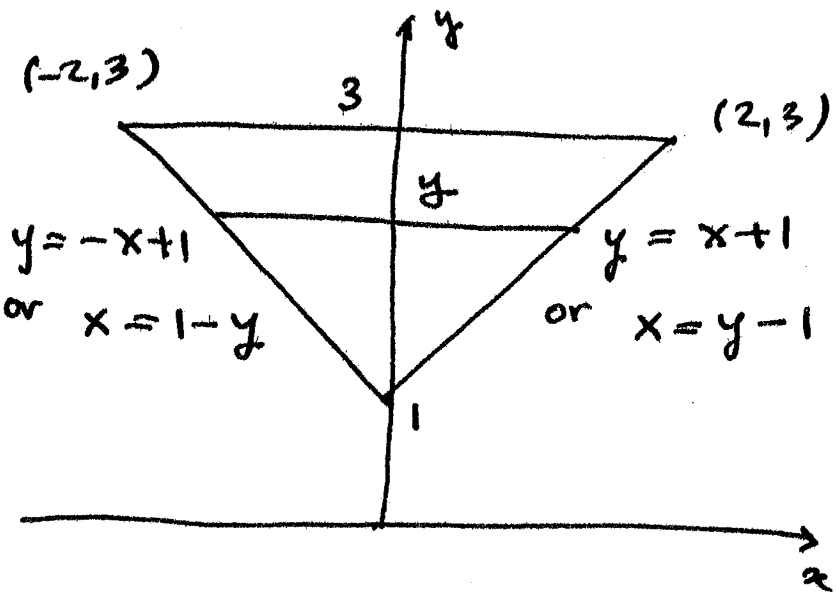
Then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

<u>Example</u> Evaluate $\iint_D (2x - y^2)\,dA$

where D is the triangular region between the lines

$$y = -x+1, \quad y = x+1, \quad y = 3$$



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The triangular region D is represented on the above picture. We see that y can be any number between 1 and 3, $1 \le y \le 3$. If we fix such y, then x can be any number between $x = 1-y$ and $x = y-1$. Thus D can be described as

$$D = \{ (x,y) \mid 1 \le y \le 3, \; 1-y \le x \le y-1 \}$$

Hence D is a domain of type II and we have

$$\iint_D (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy$$

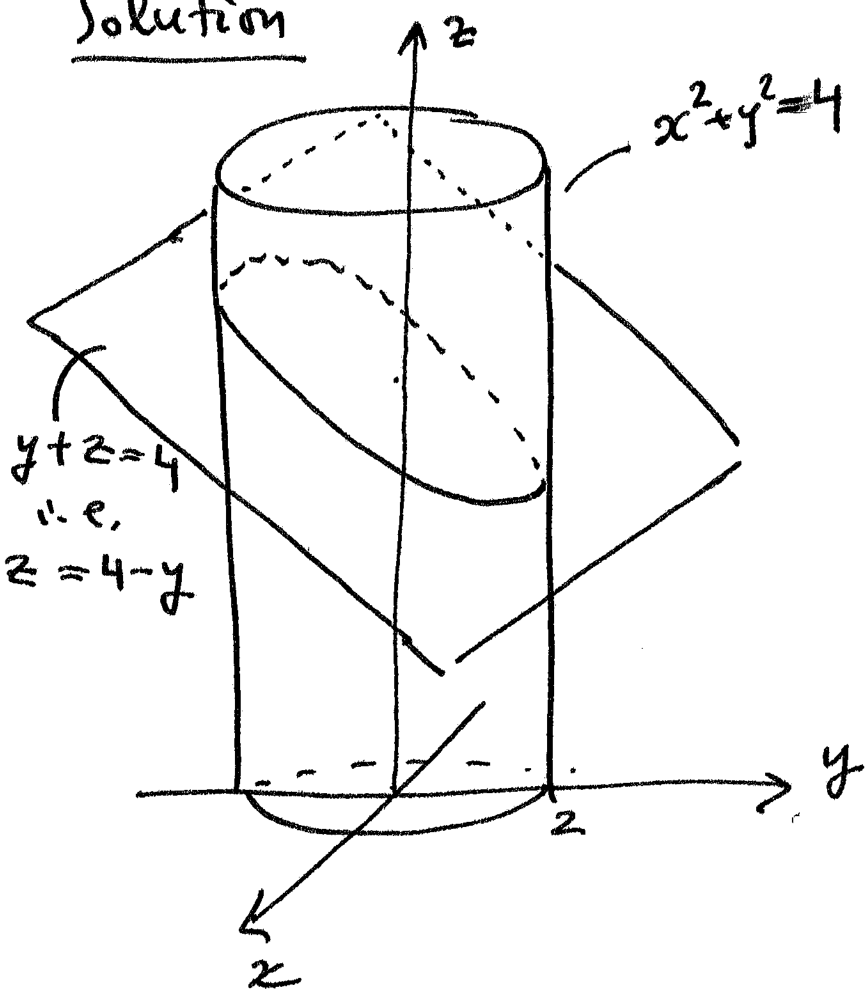
$$= \int_1^3 \left[x^2 - y^2 x \right]_{x=1-y}^{x=y-1} dy =$$

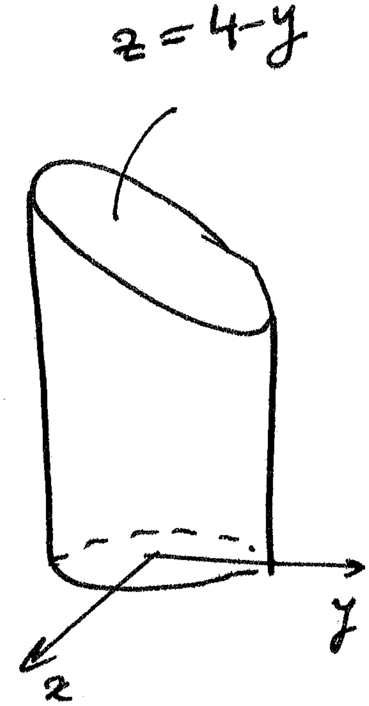
$$\int_1^3 \left[\left((y-1)^2 - y^2(y-1) \right) - \left((1-y)^2 - y^2(1-y) \right) \right] dy$$

$$= \int_1^3 2y^2 - 2y^3 \, dy = \frac{2}{3} y^3 - \frac{y^4}{2} \Big|_1^3 = -\frac{68}{3}.$$

Example Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution

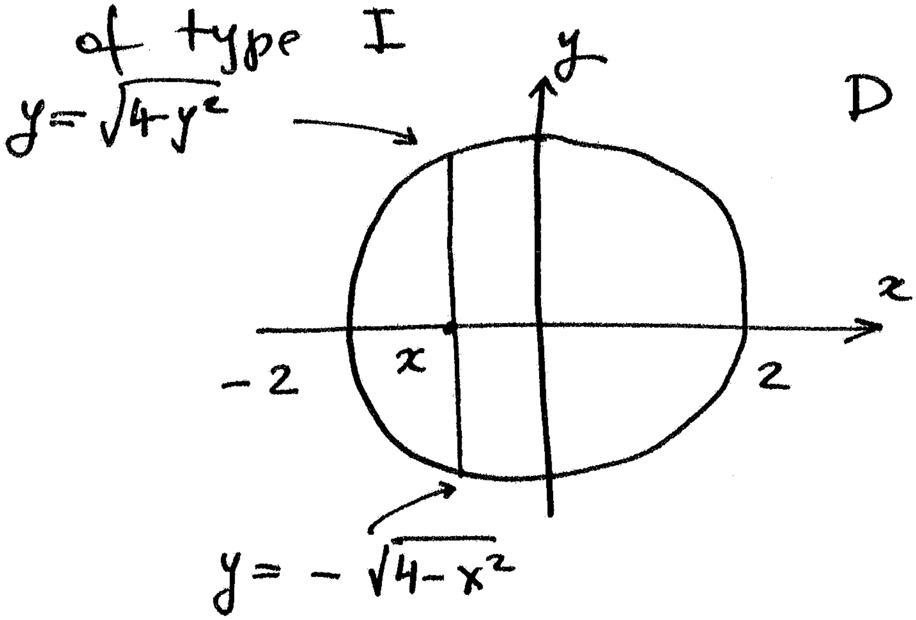




We need to find the volume under the graph of $z = 4 - y$ over the disc $D = \{ (x,y) \mid x^2 + y^2 \le 4 \}$. In other words we need to compute the integral

$$\text{Vol} = \iint_D 4 - y \, dA .$$

We represent the disc D as a domain of type I



$$D = \{ (x,y) \mid -2 \le x \le 2, \ -\sqrt{4-x^2} \le y \le \sqrt{4-y^2} \}$$

(we could also represent D as a domain of type II). We have

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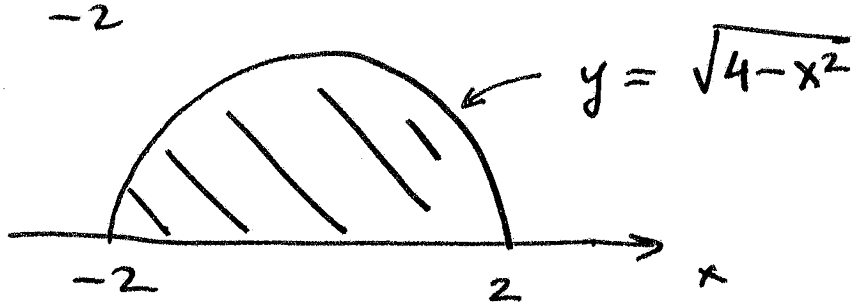
$$\iint_D 4-y\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4-y\,dy\,dx =$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx =$$

$$= \int_{-2}^{2}\left[\left(4\sqrt{4-x^2} - \frac{4-x^2}{2}\right) - \left(-4\sqrt{4-x^2} - \frac{4-x^2}{2}\right)\right] dx$$

$$= 8\int_{-2}^{2} \sqrt{4-x^2}\,dx = ?$$



$$\text{Area} = \frac{1}{2}\pi \cdot 2^2 = 2\pi$$

On the other hand

$$\text{Area} = \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

Hence

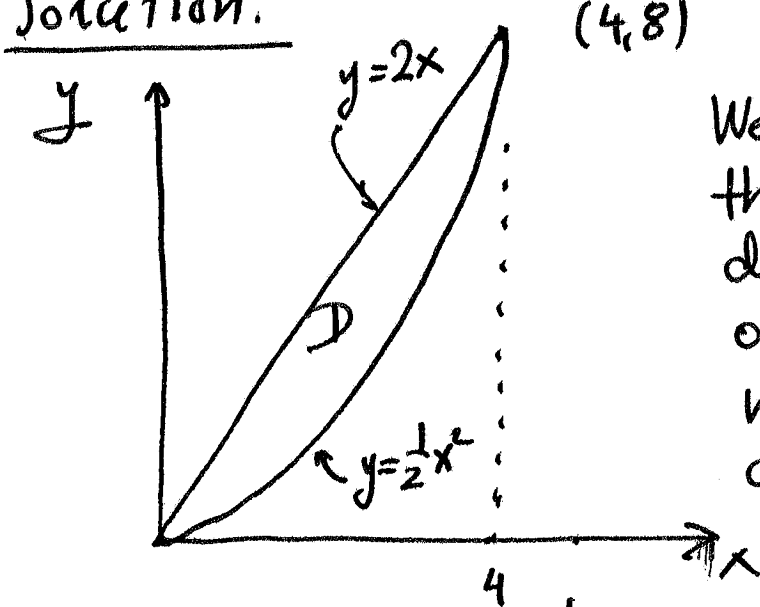
$$? = 8\int_{-2}^{2} \sqrt{4-x^2}\,dx = 8 \cdot 2\pi = \boxed{16\pi}$$

If D is a planar domain then

$$\text{Area}(D) = \iint_D 1\,dA = \iint_D dA = \iint_D dx\,dy$$

<u>Example</u> Use a double integral to find the area of the region D enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

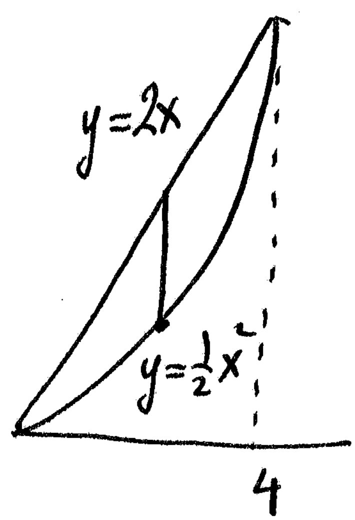
<u>Solution.</u>



We can represent this domain as a domain of type I or type II. This will give us two different methods of ~~[illegible]~~ evaluating the integral

$$\text{Area}(D) = \iint_D dA.$$

D as a domain of type I

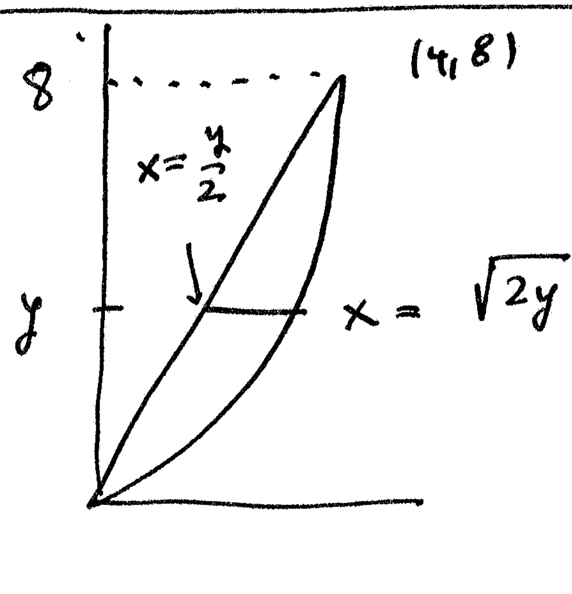


$0 \le x \le 4$
$\frac{1}{2}x^2 \le y \le 2x$

$$\text{Area}(D) = \iint_D dA = \int_0^4 \int_{\frac{1}{2}x^2}^{2x} dy\,dx =$$

$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = x^2 - \frac{x^3}{6}\Big|_0^4 = \frac{16}{3}.$$

D as a domain of type II



$0 \le y \le 8$
$\frac{y}{2} \le x \le \sqrt{2y}$

$$\text{Area}(D) = \iint_D dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{y}{2}\right) dy = \sqrt{2}\,\frac{2}{3}\, y^{3/2} - \frac{y^2}{4}\Big|_0^8 = \frac{16}{3}.$$

Example Evaluate the integral

$$I = \int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\; dx\, dy$$

by changing the order of integration.

Solution Evaluating the integral

$$\int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\; dx$$

is difficult, because it is not easy to find an antiderivative. This is why we need to change the order of integration, but what does it really mean?

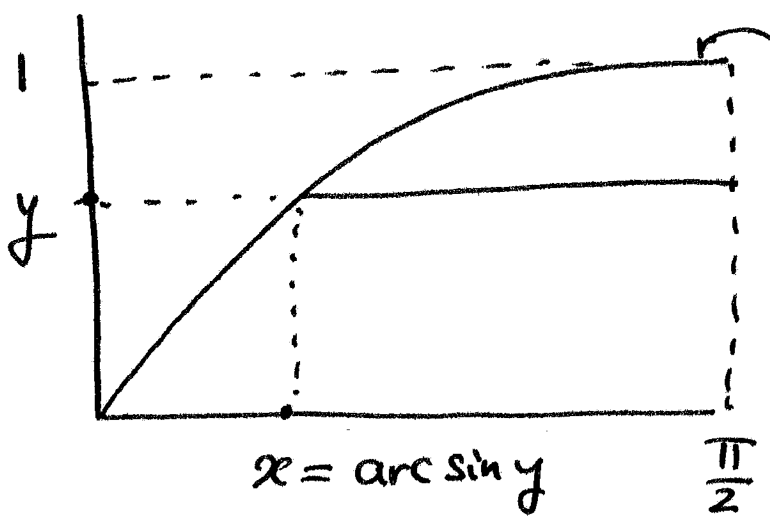
The integral I is an integral over the domain:

$D = \{(x,y) \mid 0 \le y \le 1,\ \arcsin y \le x \le \frac{\pi}{2}\}.$

This is a domain of type II.

We want to represent D as a domain of type I, then the corresponding integration will be in order $dy\,dx$.

In order to represent D as a domain of type I we need to sketch it.



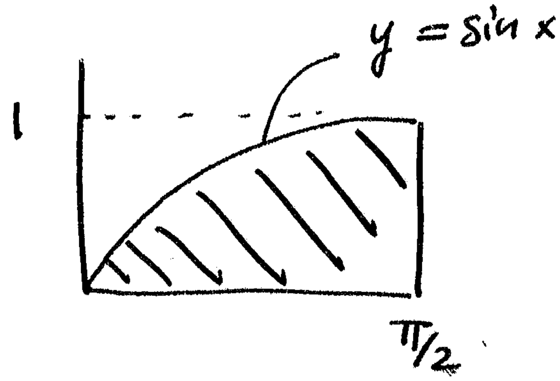
$y = \sin x$

Indeed, if $x = \arcsin y$, then $y = \sin x$

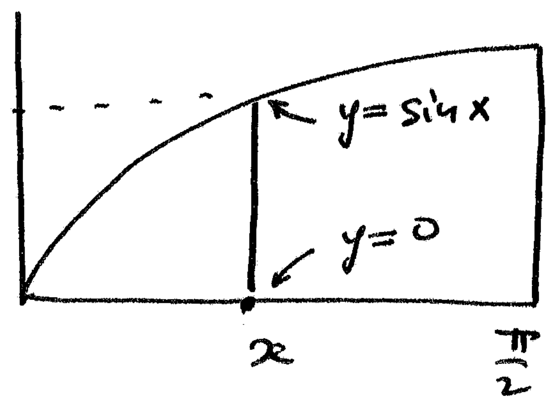
Given $0 \le y \le 1$, x changes from $x = \arcsin y$ to $x = \pi/2$

$$\arcsin y \le x \le \frac{\pi}{2}.$$

Hence we integrate over the region



We can represent this region as a domain of type I



$$0 \le x \le \frac{\pi}{2}$$
$$0 \le y \le \sin x$$

Thus

$$I = \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\, dy\, dx =$$

$$\int_0^{\pi/2} \Big[y \cos x \sqrt{1+\cos^2 x} \Big]_{y=0}^{y=\sin x} dx =$$

$$\int_0^{\pi/2} \sin x \cos x \sqrt{1+\cos^2 x}\, dx =$$

We guess the antiderivative.

Recall that $\sqrt{1+\cos^2 x} = (1+\cos^2 x)^{1/2}$.

Since the derivative lowers the exponent by 1 we should try

$$\left((1+\cos^2 x)^{3/2}\right)' = \frac{3}{2}(1+\cos^2 x)^{\frac{1}{2}}\, 2\cos x\,(-\sin x)$$

$$= -3\sin x\cos x\sqrt{1+\cos^2 x}.$$

Thus

$$\heartsuit = -\frac{1}{3}(1+\cos^2 x)^{3/2}\Big|_0^{\pi/2} =$$

$$= -\frac{1}{3}\left(1-2^{3/2}\right) = \frac{\sqrt{8}-1}{3} = \boxed{\frac{2\sqrt{2}-1}{3}}$$

<u>Example</u> Evaluate the integral

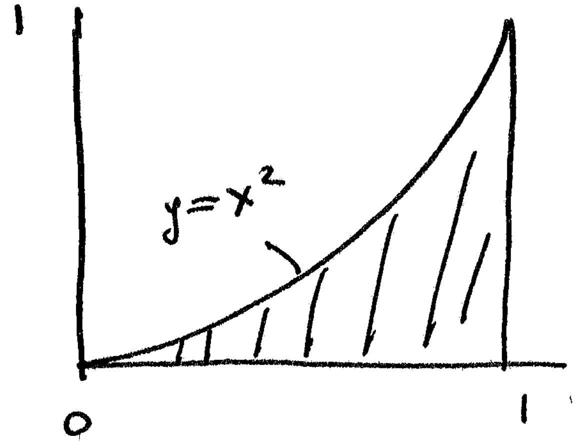
$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{y}{x}\, dx\, dy$$

by changing the order of integration.

<u>Solution</u> We integrate over

$$D = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$$

$x = \sqrt{y}$ means $y = x^2$ so the domain is (25)



This domain can be represented as

$$0 \le x \le 1$$
$$0 \le y \le x^2.$$

Thus

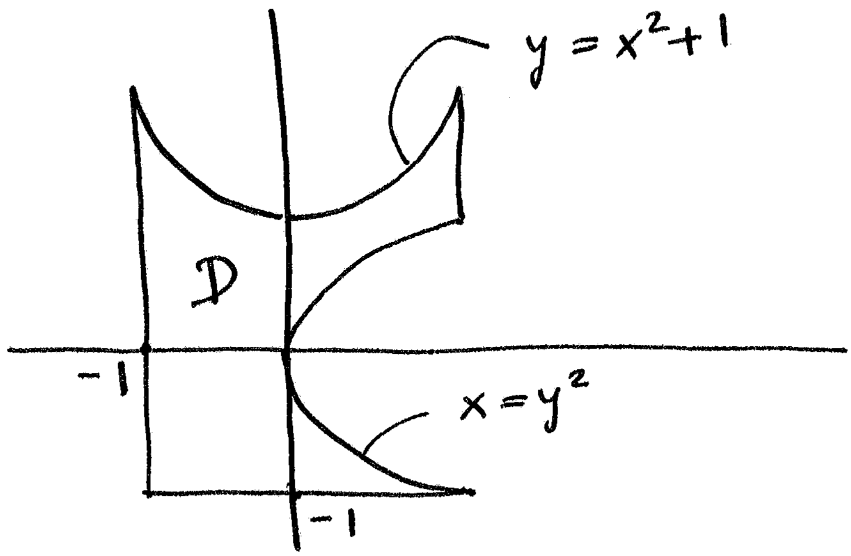
$$I = \int_0^1 \int_0^{x^2} \frac{y}{x}\, dy\, dx = \int_0^1 \frac{y^2}{2x} \Big|_{y=0}^{y=x^2} dx$$

$$= \frac{1}{2} \int_0^1 \frac{x^4}{x}\, dx = \frac{1}{2} \int_0^1 x^3 dx = \frac{x^4}{8} \Big|_0^1 = \frac{1}{8}.$$

Example Evaluate the integral

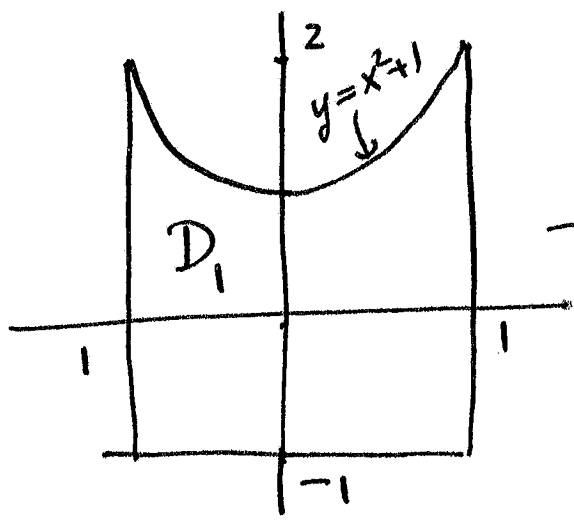
$$\iint_D xy \, dA$$

where D is described on the picture



Solution The domain D is neither of type I nor of type II, but we can represent it as a difference of two simpler domains.

D =



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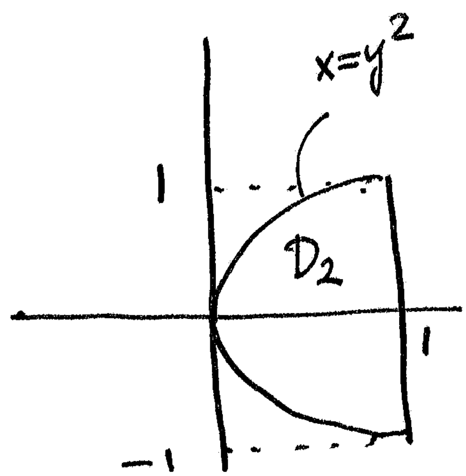
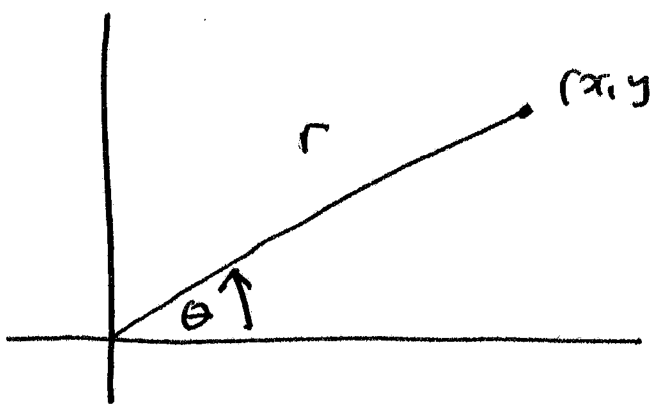


$$\iint_D xy\,dA = \iint_{D_1} xy\,dA - \iint_{D_2} xy\,dA$$

$$= \int_{-1}^{1}\int_{-1}^{x^2+1} xy\,dy\,dx - \int_{-1}^{1}\int_{y^2}^{1} xy\,dx\,dy$$

The evaluation of the integrals is left as an exercise.

Polar coordinates



The polar coordinates of a point (x,y) are (r,θ), where r is the distance to the origin and θ is the angle between the position vector $\vec{r} = \langle x,y\rangle$ and the x-axis. The angle is measured in the counterclockwise direction.

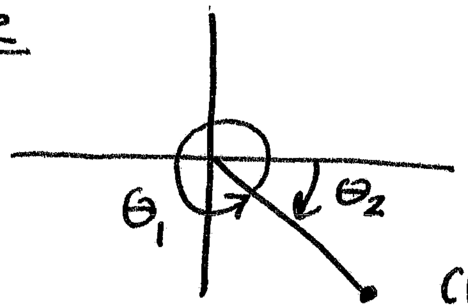
Given (r,θ) we can reconstruct the Euclidean coordinates (x,y) from the formula

$$x = r\cos\theta, \quad y = r\sin\theta .$$

Clearly, any point in $\mathbb{R}^2$ can be described by polar coordinates (r,θ), where

$$0 \leq r < \infty, \quad 0 \leq \theta < 2\pi .$$

Example



The polar coordinates of ~~[illegible]~~ $(1,-1)$ are

$$(r_1\theta_1) = \left(\sqrt{2}, 2\pi - \frac{\pi}{4}\right) = \left(\sqrt{2}, \frac{7\pi}{4}\right),$$

but also $(r_1\theta_2) = \left(\sqrt{2}, -\frac{\pi}{4}\right)$.

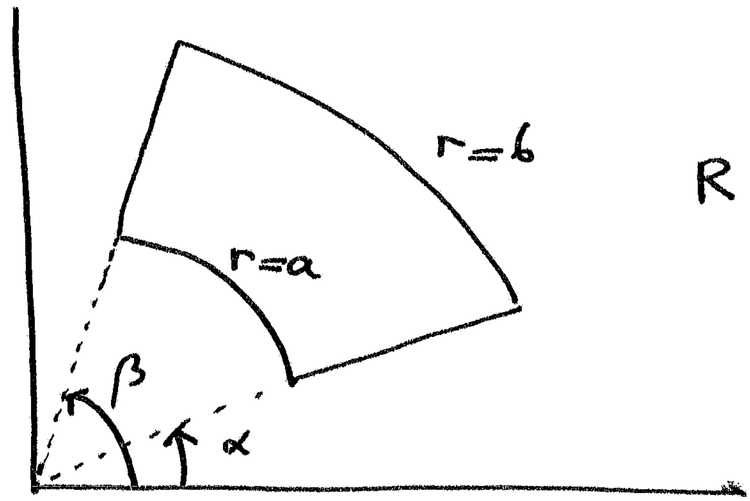
The angle θ_2 is negative because it is measured in the clockwise direction — — positive angles are measured in the counterclockwise direction.

Example ① Represent x^2+y^2 in the polar coordinates. Clearly $x^2+y^2 = r^2$ by the Phytagorean theorem. We will use this substitution very often.

② Represent xy in the polar coordinates.

$$xy = r\cos\theta \; \sin\theta = r^2 \cos\theta \sin\theta.$$

Some domains that have circular shape can be easily described in polar coordinates



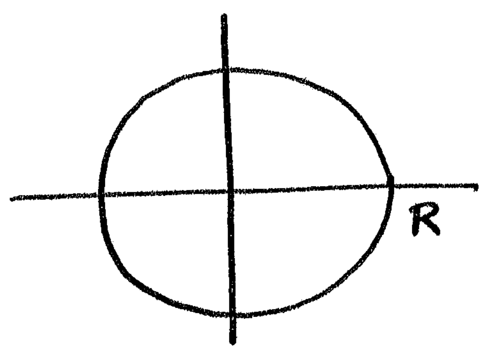
$$R = \{ (r,\theta) \mid a \le r \le b, \ \alpha \le \theta \le \beta \}$$

For obvious reasons R is called a polar rectangle.

<u>Examples of polar rectangles</u>

①

$$\left\{ (r,\theta) \;\middle|\; \begin{matrix} 0 \le r \le R \\ 0 \le \theta \le 2\pi \end{matrix} \right\}$$



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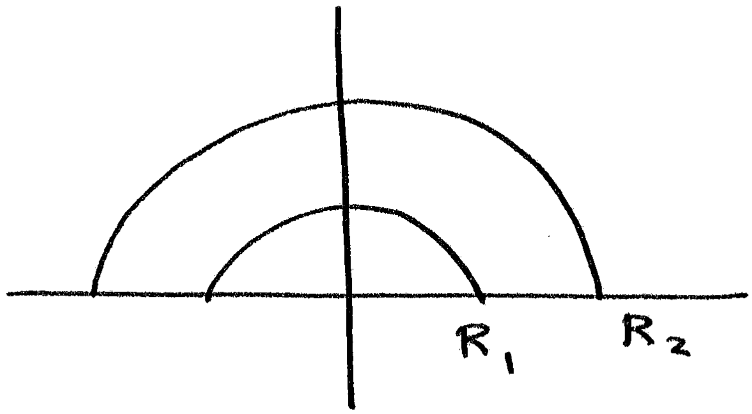
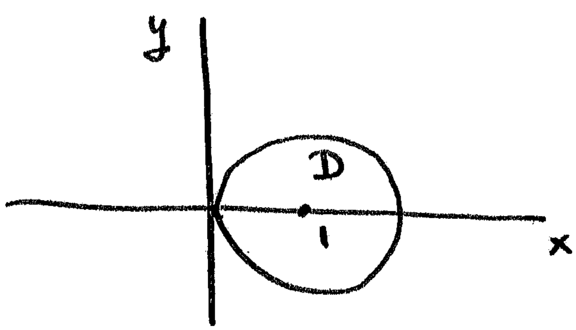


$$\{ (r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le \pi \}$$

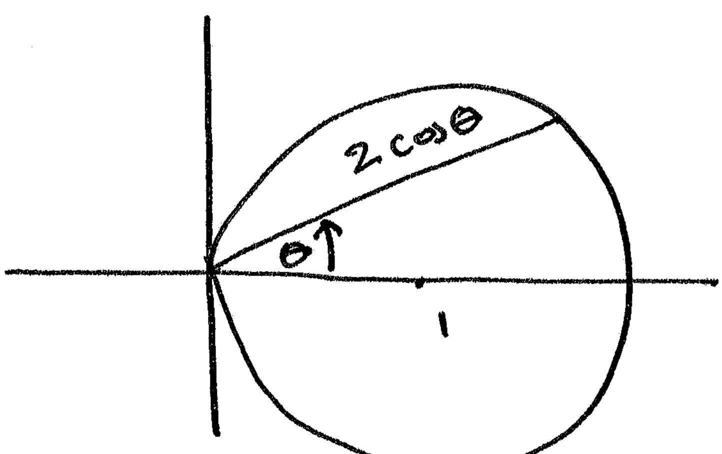
<u>Example</u> Describe the disc

$$D = \{ (x,y) \mid (x-1)^2 + y^2 \le 1 \}$$

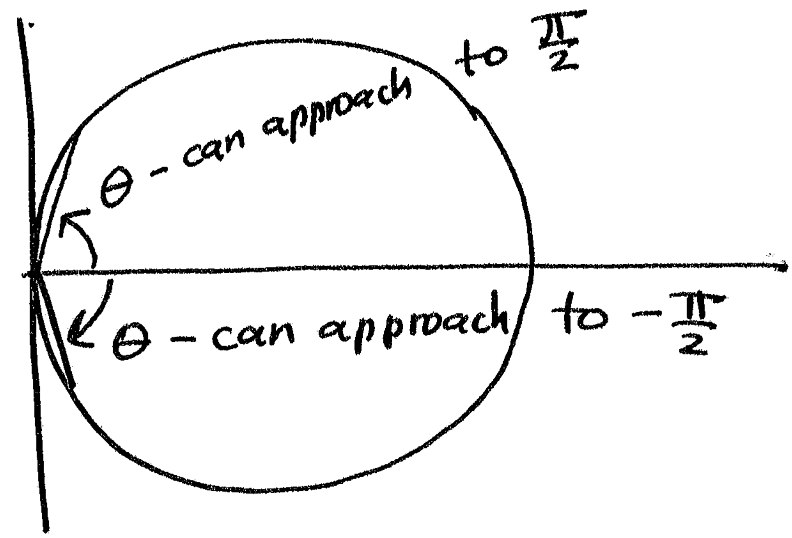
in polar coordinates.



$$(x-1)^2 + y^2 \le 1$$
$$x^2 - 2x + 1 + y^2 \le 1$$
$$x^2 + y^2 \le 2x$$
$$r^2 \le 2r\cos\theta$$
$$\boxed{r \le 2\cos\theta}$$



What is the range of θ?



Thus the disc $D = \{(x,y) \mid (x-1)^2 + y^2 \leq 1\}$ in polar coordinates is described as

$$D = \left\{(r,\theta) \;\middle|\; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\right\}.$$

It is important to understand this example. Such examples will show up in problems.

Integration in polar coordinates is described in the following theorem

<u>Theorem</u> If $R = \{(r,\theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$ then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

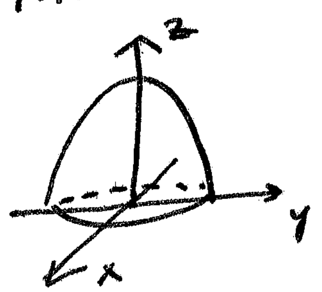
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

In the integration in polar coordinates we substitute

$$x \longmapsto r\cos\theta$$
$$y \longmapsto r\sin\theta$$
$$dx\,dy \longmapsto r\,dr\,d\theta$$

Example Find the volume of the solid bounded by the plane $z=0$ and the surface $z = 1-x^2-y^2$.

Solution. The surface intersects with the plane $z=0$ along the circle $0=1-x^2-y^2$ i.e. $x^2+y^2=1$. Thus we are asked to find the volume under the graph of



$z = 1-x^2-y^2$ over the disc $D = \{(x,y) \mid x^2+y^2 \leq 1\}$.
In other words we are asked to compute the integral

$$\iint_D 1-x^2-y^2\,dx\,dy.$$

In polar coordinates

$$D = \{(r,\theta) / 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

and

$$z = 1-x^2-y^2 = 1-r^2.$$

Thus

$$\iint_D 1-x^2-y^2\,dx\,dy = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r-r^3\,dr\,d\theta \underset{\star}{=} \int_0^{2\pi} d\theta \int_0^1 r-r^3\,dr$$

$$= 2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}.$$

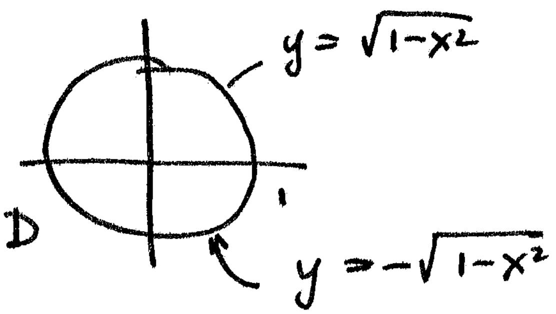
Remark 1 The equality $\star$ requires some explanations. In general we have

$$\int_a^b\int_c^d f(r)g(\theta)\,dr\,d\theta = \int_a^b g(\theta)\,d\theta \int_c^d f(r)\,dr.$$

In particular

$$\int_a^b\int_c^d f(r)\,dr\,d\theta = \int_a^b d\theta \int_c^d f(r)\,dr.$$

Remark 2 We could attempt to compute the integral without using polar coordinates. In Euclidean coordinates the disc is



$$D = \left\{(x,y) \,\middle|\, \begin{array}{l} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{array}\right\}$$

Hence the integral becomes

$$\iint_D 1-x^2-y^2\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\,dx$$

and this integral is not easy to compute.

Example Evaluate the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$

Solution We integrate over the upper half-disc

In polar coordinates

$$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi\}$$

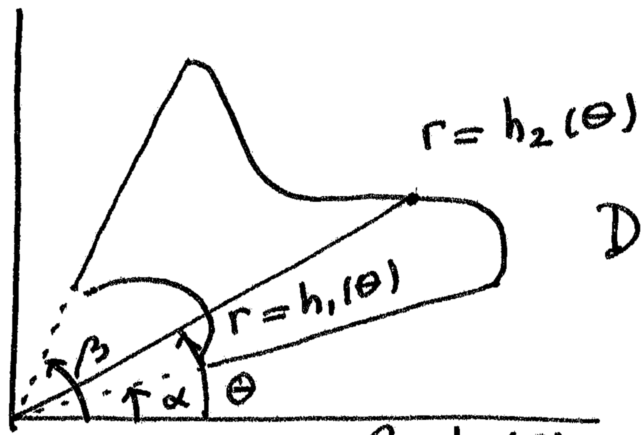
Hence

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dx\,dy = \iint_D e^{x^2+y^2}\,dx\,dy =$$

$$= \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi} d\theta \int_0^1 e^{r^2}\,r\,dr =$$

$$\pi \cdot \left.\frac{e^{r^2}}{2}\right|_0^1 = \frac{\pi(e-1)}{2}.$$

We can integrate in polar coordinates over more complicated domains than the polar rectangles. The following domain is a polar coordinate counterpart of a domain of type I



$$D = \left\{ (r,\theta) \;\middle|\; \begin{matrix} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{matrix} \right\}$$

$$\iint_D f(x,y)\,dx\,dy = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta .$$

<u>Example</u> Evaluate the integral

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$

by converting it to polar coordinates.

<u>Solution</u> First we have to find out over which domain we integrate.

$0 \le x \le 2, \quad 0 \le y \le \sqrt{2x-x^2}$

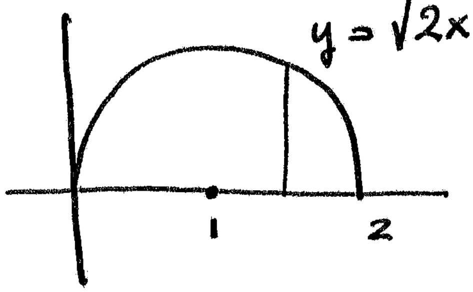
$$y = \sqrt{2x - x^2}$$

$$y^2 = 2x - x^2$$

$$(*) \qquad x^2 + y^2 = 2x$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1 \quad \text{– a circle}$$



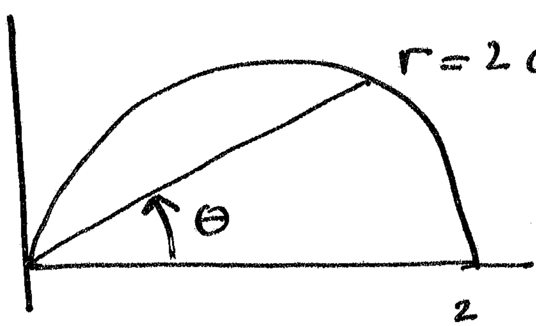
x changes from 0 to 2

y changes from 0 to $\sqrt{2x-x^2}$

In polar coordinates the equation $(*)$ is

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$



$$D = \left\{ (r, \theta) \,\middle|\, \begin{array}{l} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{array} \right\}$$

Hence

$$I = \int_0^{\pi/2} \int_0^{2\cos\theta} \underbrace{r\sin\theta}_{y} \sqrt{\underbrace{r^2}_{x^2+y^2}} \, \underbrace{r\,dr\,d\theta}_{dA} =$$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} \sin\theta\, r^3\,dr\,d\theta =$$ (37)

$$\int_0^{\pi/2} \sin\theta \left.\frac{r^4}{4}\right|_{r=0}^{r=2\cos\theta} d\theta = \int_0^{\pi/2} \sin\theta\, \frac{(2\cos\theta)^4}{4}\, d\theta =$$

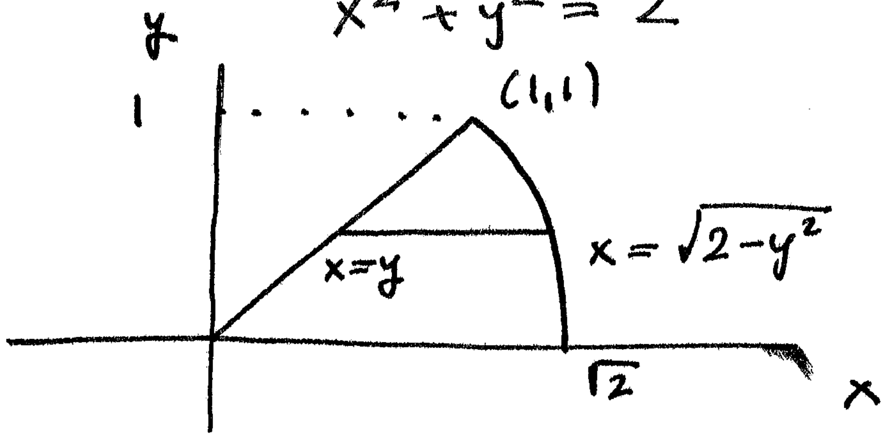
$$4\int_0^{\pi/2} \sin\theta\,(\cos\theta)^4\,d\theta = 4\left[-\frac{(\cos\theta)^5}{5}\right]_0^{\pi/2} = \frac{4}{5}.$$

<u>Example</u> Evaluate the integral

$$I=\int_0^1\int_y^{\sqrt{2-y^2}} x+y\;dx\,dy$$

<u>Solution</u> $0\le y\le 1,\quad y\le x\le\sqrt{2-y^2}$

$$x=\sqrt{2-y^2}$$
$$x^2=2-y^2$$
$$x^2+y^2=2$$



In polar coordinates the domain is described by

$$D = \{ (r, \theta) \mid 0 \le \theta \le \frac{\pi}{4},\ 0 \le r \le \sqrt{2} \}$$

Hence

$$I = \int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta =$$

$$= \int_0^{\pi/4} (\cos\theta + \sin\theta)\, d\theta \int_0^{\sqrt{2}} r^2\, dr =$$

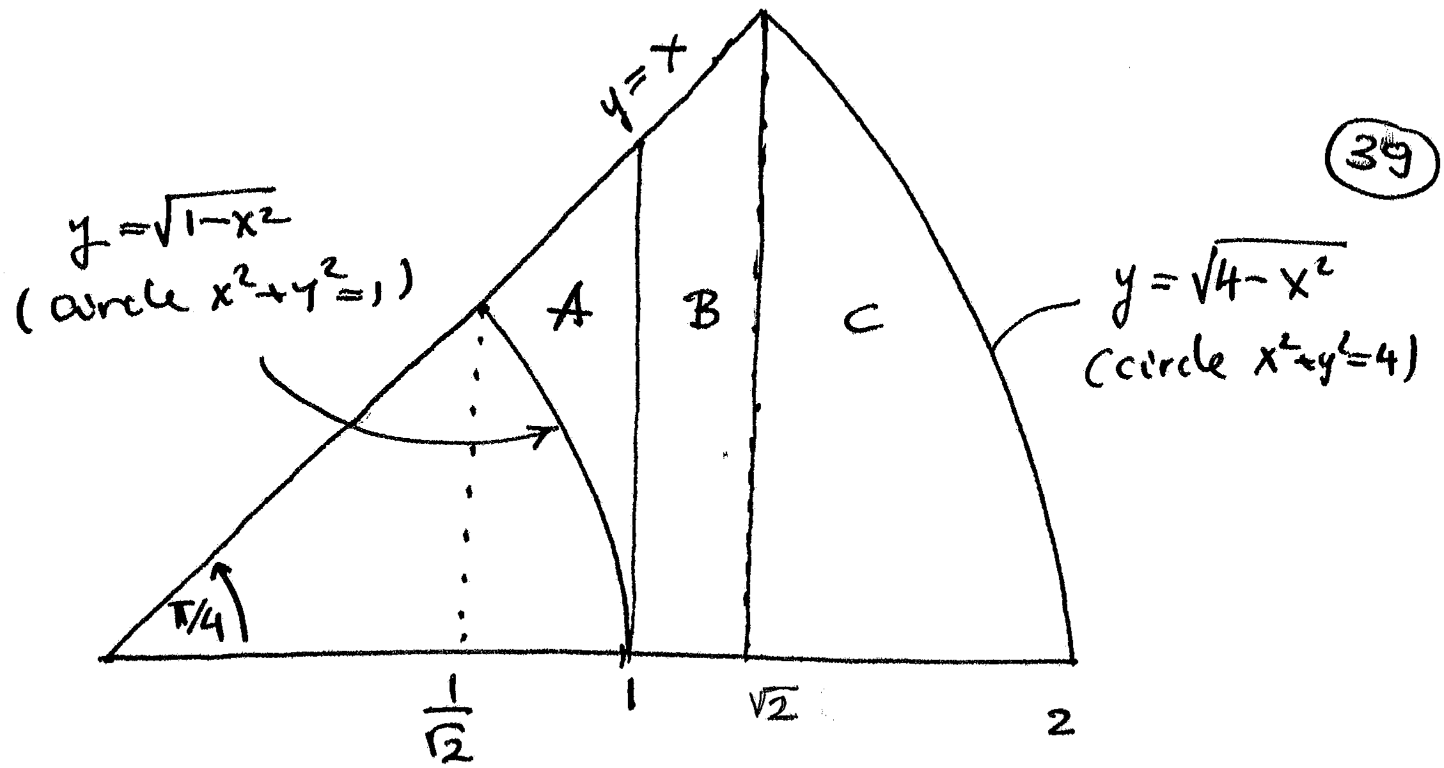
$$= \Big[\sin\theta - \cos\theta \Big]_0^{\pi/4} \Big[\frac{r^3}{3} \Big]_0^{\sqrt{2}} =$$

$$\Big[\Big(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \Big) - (0-1) \Big] \cdot \Big[\frac{(\sqrt{2})^3}{3} \Big] = \frac{2\sqrt{2}}{3}.$$

__Example__ Evaluate the integral

$$I = \int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_1^{\sqrt{2}} \int_0^{x} xy\, dy\, dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy\, dy\, dx.$$

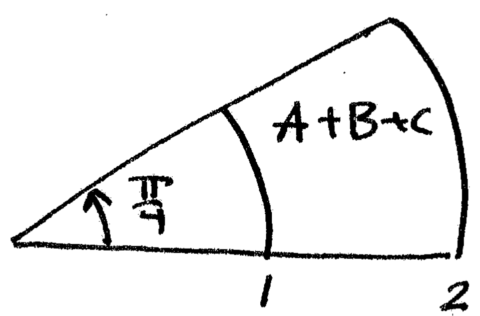
__Solution__ As we will see the three domains add up to one that has a simple description in polar coordinates.



$$A = \left\{ (x,y) \,\middle|\, \frac{1}{\sqrt{2}} \le x \le 1 ,\ \sqrt{1-x^2} \le y \le x \right\}$$

$$B = \left\{ (x,y) \,\middle|\, 1 \le x \le \sqrt{2},\ 0 \le y \le x \right\}$$

$$C = \left\{ (x,y) \,\middle|\, \sqrt{2} \le x \le 2 ,\ 0 \le y \le \sqrt{4-x^2} \right\}$$



$$A+B+C = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le \pi/4 \end{array} \right\}$$

$$I = \iint_A xy\,dy\,dx + \iint_B xy\,dy\,dx + \iint_C xy\,dy\,dx$$

$$= \iint_{A+B+C} xy\,dy\,dx = \int_0^{\pi/4}\int_1^2 \underbrace{r\cos\theta}_{x}\ \underbrace{r\sin\theta}_{y}\ \underbrace{r\,dr\,d\theta}_{dx\,dy} =$$

$$= \int_0^{\pi/4} \sin\theta \cos\theta \, d\theta \int_1^2 r^3 dr =$$

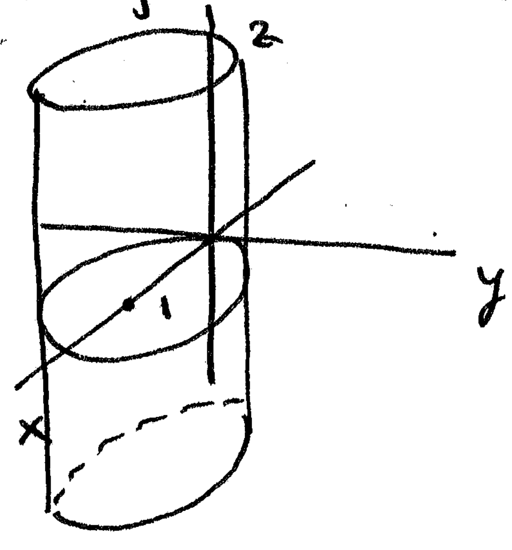
$$= \frac{\sin^2\theta}{2}\Big|_0^{\pi/4} \; \frac{r^4}{4}\Big|_1^2 = \left(\frac{(\sqrt{2}/2)^2}{2} - 0\right)\left(4 - \frac{1}{4}\right) = \frac{15}{16}.$$

Important:
Remember the formulas:

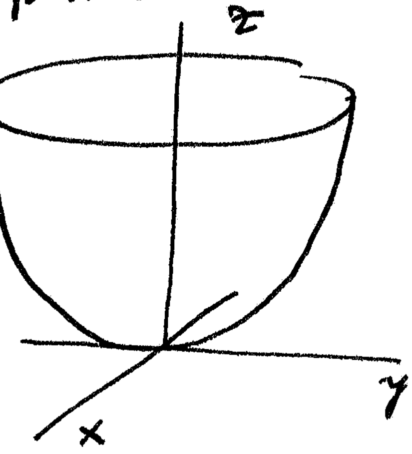
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Example Find the volume of the solid under the paraboloid $z = x^2+y^2$, above the xy plane and inside the cylinder $x^2+y^2 = 2x$.

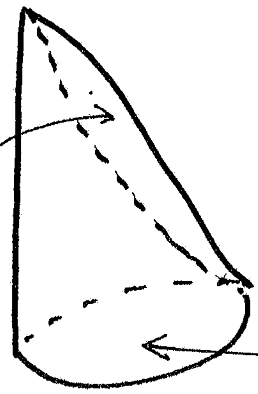
Solution The cylinder
$x^2+y^2 = 2x$ or $(x-1)^2+y^2 = 1$



The paraboloid $z = x^2+y^2$



Domain: under the paraboloid, inside the cylinder, above the xy-plane



graph of $z = x^2+y^2$

← we need to find volume

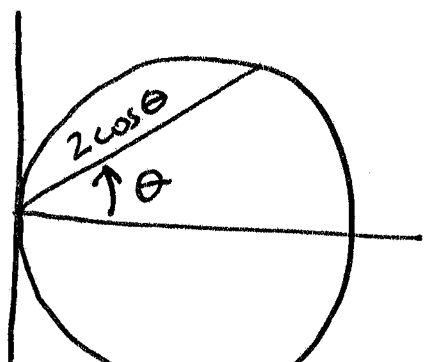
$D = \{(x,y) \mid x^2+y^2=2x\}$

Thus we need to compute the integral

$$V = \iint_D x^2+y^2 \, dA$$

D in polar coordinates:

$$x^2+y^2 = 2x$$
$$r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$



$$D = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{array} \right\}$$

(see pp. 30–31).

$$V = \iint_D x^2+y^2 dA = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} \underbrace{r^2 \, r}_{r^3} \, dr\, d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left. \frac{r^4}{4} \right|_{r=0}^{r=2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16\cos^4\theta}{4}\, d\theta$$

$$= 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta \underset{\text{even function}}{=} 8 \int_0^{\pi/2} \cos^4\theta\, d\theta = \heartsuit$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\heartsuit = 8 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = 2 \int_0^{\pi/2} (1+\cos 2\theta)^2\, d\theta$$

$$= 2 \int_0^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta =$$

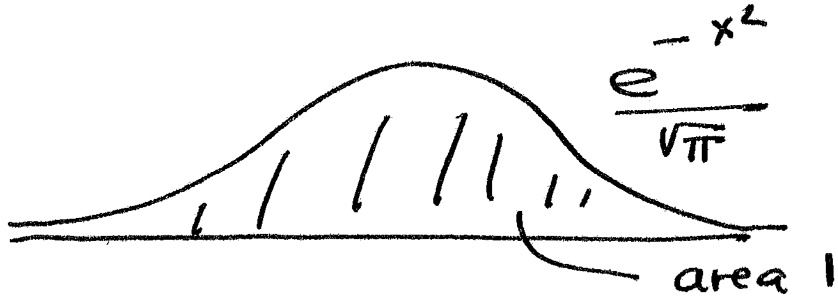
$$= 2 \int_0^{\pi/2} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\, d\theta =$$

$$2\left[\theta + \sin 2\theta + \frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]_0^{\pi/2} = \frac{3\pi}{2}.$$

As an application of integration in polar coordinates we will prove the following important and beautiful

Theorem $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Hence the area under the graph of $\frac{e^{-x^2}}{\sqrt{\pi}}$ equals 1. This function is called the Gauss normal distribution



The normal distribution plays a fundamental role in statistics.
The area under the graph represents the total probability and hence must be equal 1.
There is no simple proof of this theorem, because there is no formula

for the antiderivative of e^{-x^2}. The theorem is surprising. We integrate e^{-x^2} and the answer is $\sqrt{\pi}$.

<u>Proof</u> Let

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy.$$

This is an improper integral. We have

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2.$$

the integrals are equal

On the other hand

$$I = \iint_{R^2} e^{-(x^2+y^2)}dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}r\,dr = 2\pi \int_0^{\infty} e^{-r^2}r\,dr$$

$$\int_0^{\infty} e^{-r^2}r\,dr = \lim_{R\to\infty} \int_0^{R} e^{-r^2}r\,dr =$$

this is how we define the improper integral

$$= \lim_{R\to\infty} -\frac{e^{-r^2}}{2}\Big|_0^R =$$

$$= \lim_{R\to\infty} \left(-\underbrace{\frac{e^{-R^2}}{2}}_{\to 0} + \frac{e^0}{2} \right) = \frac{1}{2}.$$

Hence $I = 2\pi \cdot \frac{1}{2} = \pi.$

We have

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = I = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \qquad \square$$

Tripple integrals

The tripple integrals are defined through the Riemann sum approximation almost in the same way as double integrals. The double integrals $\iint_D f(x,y)\,dA$ can be interpreted as a volume of the solid under the graph of f, while tripple integrals $\iiint_D f(x,y,z)\,dV$ would need to be interpreted as a four dimensional volume of a four dimensional solid under the graph of f. It is not entirely clear what it means and for this reason we

will not refer to this interpretation.
There are however, purely 3 dimensional interpretations of the tripple integral.

If D is a 3D solid in $\mathbb{R}^3$, then
$Vol(D) = \iiint_D dV$ is the volume of D

If $\sigma(x,y,z)$ is a mass density of a solid D in $\mathbb{R}^3$ (measured e.g. in kg/m^3) then
$\iiint_D \sigma(x,y,z)\,dV$ is the total mass of D.

The integral over a 3 dimensional rectangular domain can be computed as an iterated integral

<u>Theorem (Fubini)</u>

If $R = [a,b] \times [c,d] \times [e,k] =$
$= \{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ e \le z \le k\}$
and $f(x,y,z)$ is a continuous function defined on R, then

$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^k f(x,y,z)\,dz\,dy\,dx$$

$$= \int_e^k \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \int_a^b \int_e^k \int_c^d f(x,y,z)\, dy\, dz\, dx$$

$$= \ldots$$

$= \ldots$ means that there are three more ways to represent the integral with the order of integration corresponding to

$$dx\, dy\, dz, \quad dx\, dz\, dy, \quad dz\, dx\, dy .$$

<u>Example</u>

$$\int_0^3 \int_{-1}^2 \int_0^1 x y z^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \frac{x^2 y z^2}{2}\Big|_{x=0}^{x=1} dy\, dz =$$

$$\int_0^3 \int_{-1}^2 \frac{y z^2}{2}\, dy\, dz = \int_0^3 \frac{y^2 z^2}{4}\Big|_{y=-1}^{y=2} dz =$$

$$\int_0^3 \frac{4z^2}{4} - \frac{z^2}{4}\, dz = \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\Big|_0^3 = \frac{27}{4}.$$

Computing this integral with different order of integration (5 more orders) will always give the same answer – Fubini theorem.

If a region E in $\mathbb{R}^3$ is defined by

$$E = \{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \underbrace{\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz}_{\substack{\text{the value of this} \\ \text{integral depends on} \\ (x,y) \text{ only.}}}\,dx\,dy$$

Hence the tripple integral is reduced to a double integral over D.

<u>Example</u> Write the integral

$$\iiint_E f(x,y,z)\,dV$$

where

$$E = \{ (x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y) \}$$

as an iterated integral.

<u>Solution</u>

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Indeed, we can describe the domain E as follows:

$$E = \{(x,y,z) \mid (x,y) \in D, \quad u_1(x,y) \le z \le u_2(x,y)\}$$

where $D = \{(x,y) \mid a \le x \le b, \; g_1(x) \le y \le g_2(x)\}$.

Hence

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy = \heartsuit$$

$$\iint_D \dots dx\,dy \;=\; \int_a^b \int_{g_1(x)}^{g_2(x)} \dots dy\,dx$$

D is a domain of type I

$$\heartsuit = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx .$$

<u>Example</u>

Let $E = \{(x,y,z) \mid (x,y) \in D, \; 0 \le z \le f(x,y)\}$ be the solid under the graph of a positive continuous function $z = f(x,y)$. Then we can compute $Vol(E)$ using two different methods

① $Vol(E) = \iint_D f(x,y)\,dx\,dy$

② $Vol(E) = \iiint_E dx\,dy\,dz$

As we will see the second method will lead to the same integral as in ①. Indeed, using the formula from p. 48 we have

$$vol(E) = \iiint_E 1\,dx\,dy\,dz = \iint_D \underbrace{\int_0^{f(x,y)} 1\,dz}_{f(x,y)}\,dx\,dy$$

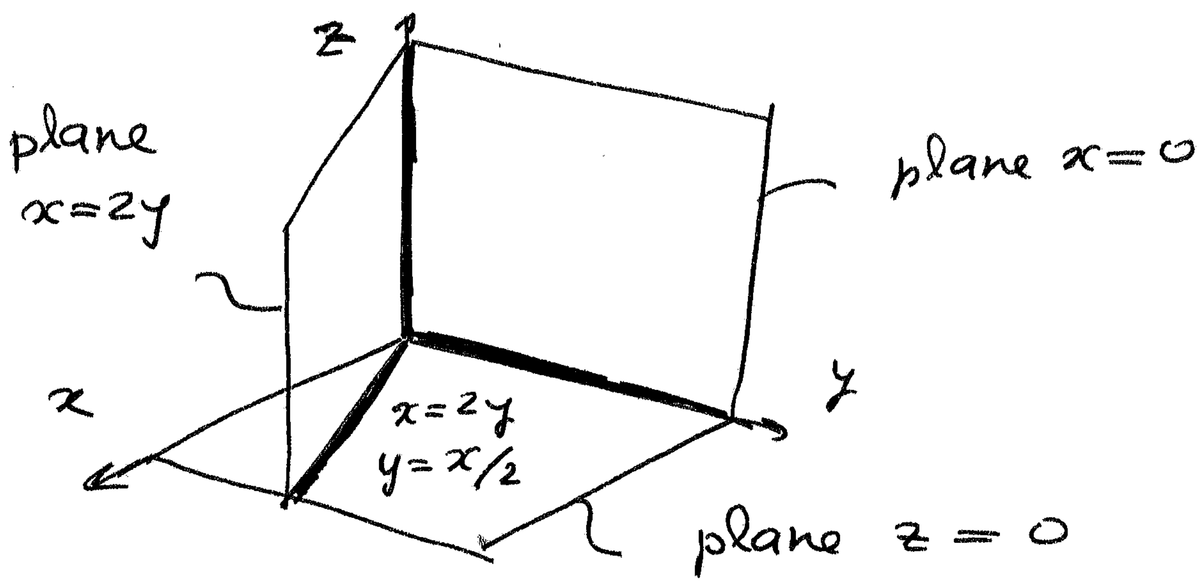
$$= \iint_D f(x,y)\,dx\,dy .$$

<u>Example</u> Find volume of the tetrahedron T bounded by the planes

$x+2y+z=0$, $x=2y$, $x=0$ and $z=0$

<u>Solution</u> Three of the four faces are contained in the planes

$x=0$, $z=0$ and $x=2y$



The fourth face is contained in the plane

$$x + 2y + z = 2$$

and we want to find out how this plane intersects with the other three planes

$$x = 0, \quad z = 0 \quad \text{and} \quad x = 2y.$$

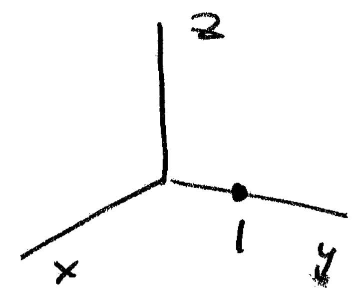
To find this we need to find points at which the plane $x + 2y + z = 2$ intersects with

- y - axis
- z - axis
- line $x = 2y$ in the xy plane

These are the bold lines on the picture on p. 50.

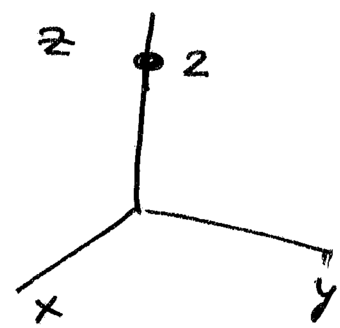
<ins>y - axis</ins> $\quad x = z = 0$

$$x + 2y + z = 2$$
$$0 + 2y + 0 = 2$$
$$y = 1$$



<ins>z - axis</ins> $\quad x = y = 0$

$$x + 2y + z = 2$$
$$0 + 2 \cdot 0 + z = 2$$
$$z = 2$$



line $x = 2y$ or $y = \frac{x}{2}$

The line is in the xy plane

So $z = 0$

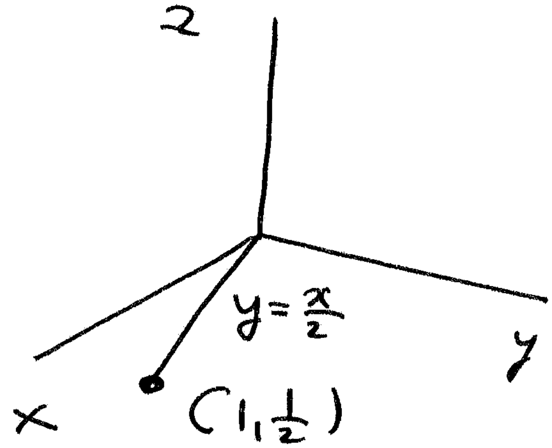
$$x + 2y + z = 2$$

$$x + 2\left(\frac{x}{2}\right) + 0 = 2$$

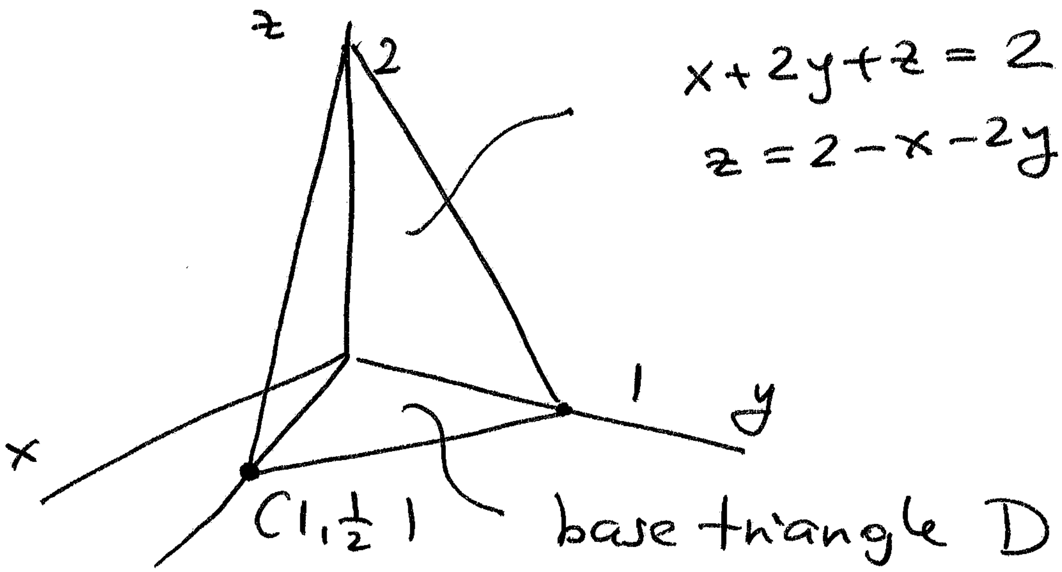
$$2x = 2$$

$$x = 1$$

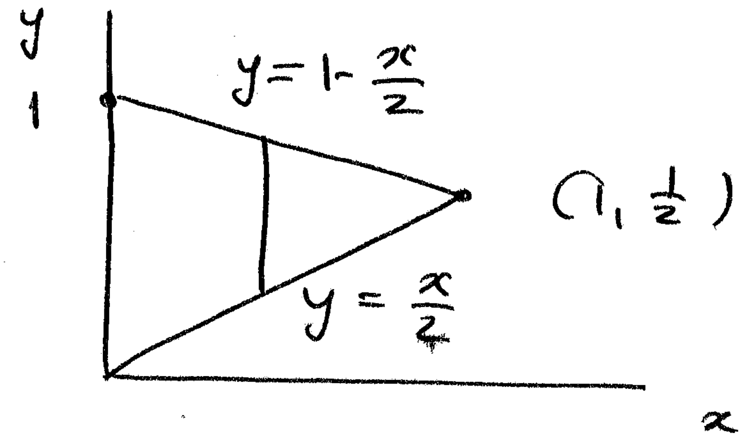
$$y = \frac{x}{2} = \frac{1}{2}$$



Thus the plane $x + 2y + z = 2$ intersects with the other three planes as shown on the picture



Triangle D



The top side of the triangle D, line connecting 1 on the y-axis with $(1, \frac{1}{2})$ has the y-intercept 1 and the slope equal $-\frac{1}{2}$ so

$$y = 1 - \frac{1}{2} x.$$

Thus

$$D = \{ (x,y) \mid 0 \le x \le 1, \ \frac{x}{2} \le y \le 1 - \frac{x}{2} \}$$

Hence

$$T = \{ (x,y,z) \mid (x,y) \in D, \ 0 \le z \le 2 - x - 2y \}$$

$$= \left\{ (x,y,z) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ \frac{x}{2} \le y \le 1 - \frac{x}{2} \\ 0 \le z \le 2 - x - 2y \end{array} \right\}$$

So

$$\text{vol}(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1 - x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

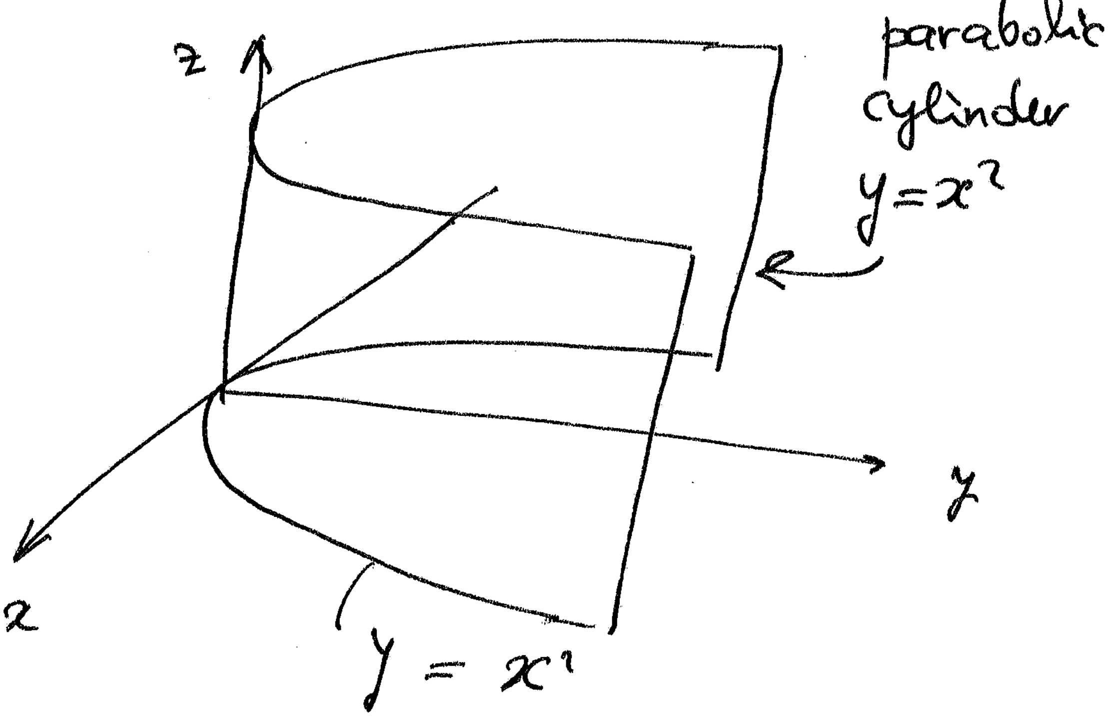
$$= \int_0^1 \int_{x/2}^{1-x/2} 2 - x - 2y \, dy \, dx$$

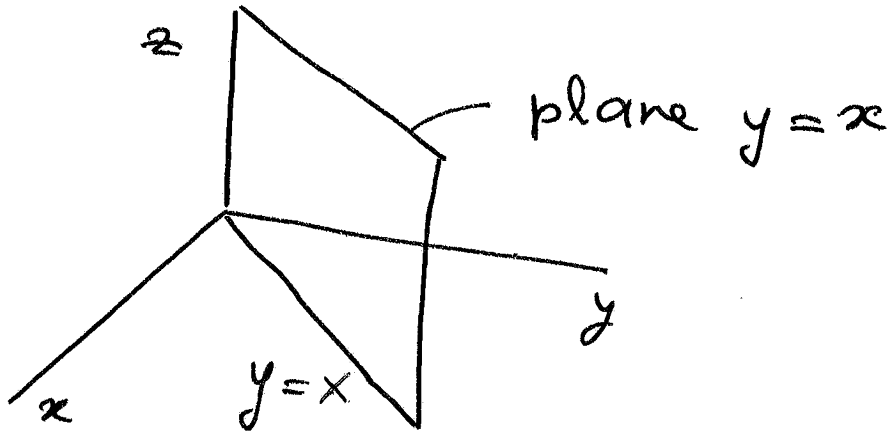
$$= \int_0^1 \Big[2y - xy - y^2 \Big]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} \, dx$$

$$= \int_0^1 x^2 - 2x + 1 \, dx = \frac{x^3}{3} - x^2 + x \Big|_0^1 = \frac{1}{3}$$

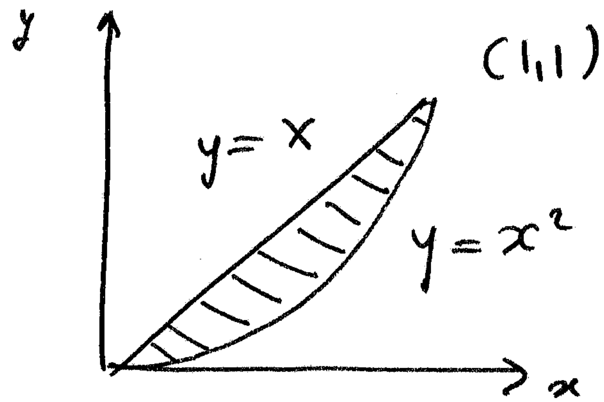
Example Evaluate $\iiint_E (x+2y)\, dV$ where E is bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$ and $z = 0$

Solution



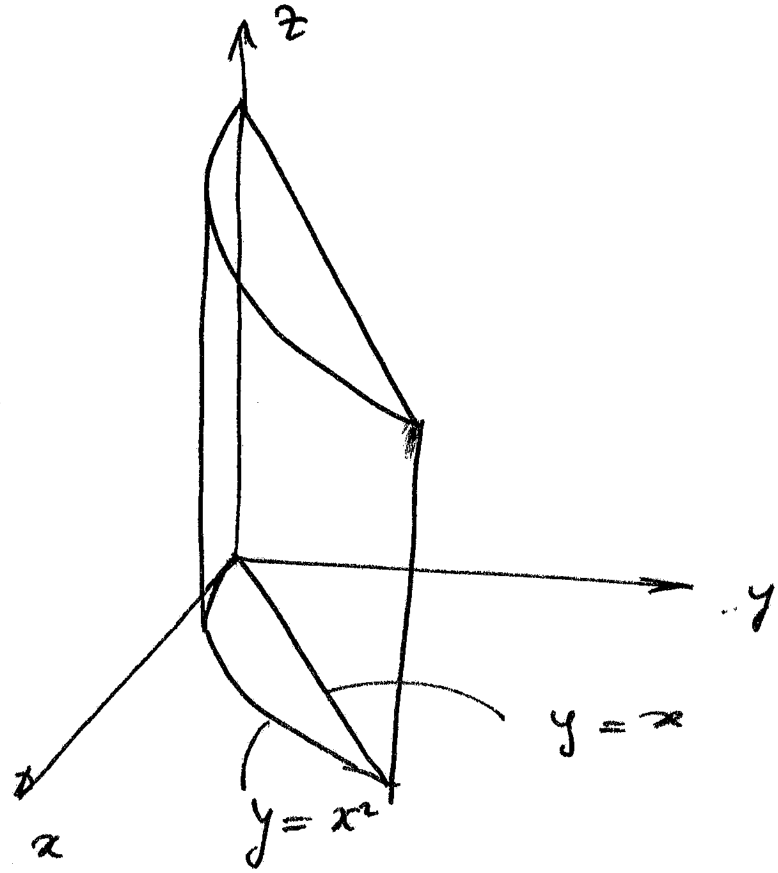


On the xy plane the line $y=x$ and the parabola $y=x^2$ bound the region



$$\{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\}$$

So the solid bounded by the cylinder $y=x^2$ and the plane looks as follows:



Now z is between the planes $z = 0$ and $z = x$. Hence

$E = \{(x,y,z) | 0 \le x \le 1, x^2 \le y \le x, 0 \le z \le x\}$

thus

$$\iiint_E (x+2y)\,dV = \int_0^1 \int_{x^2}^{x} \int_0^{x} (x+2y)\,dz\,dy\,dx =$$

$$\int_0^1 \int_{x^2}^{x} (x+2y)z \Big|_{z=0}^{z=x} dy\,dx =$$

$$\int_0^1 \int_{x^2}^{x} x^2+2yx \, dy \, dx =$$

$$\int_0^1 x^2y + y^2x \Big|_{y=x^2}^{y=x} dx =$$

$$\int_0^1 \left[(x^3+x^3) - (x^4+x^5) \right] dx =$$

$$\int_0^1 2x^3 - x^4 - x^5 dx = \frac{x^4}{2} - \frac{x^5}{5} - \frac{x^6}{6} \Big|_0^1 =$$

$$= \frac{1}{2} - \frac{1}{5} - \frac{1}{6} = \frac{2}{15}$$

Example Rewrite the integral from the previous example

$$\iiint_E (x+2y)\,dV = \int_0^1 \int_{x^2}^{x} \int_0^{x} (x+2y)\,dz\,dy\,dx$$

as

$$\int_?^? \int_?^? \int_?^? (x+2y)\,dy\,dx\,dz$$

and evaluate.

Solution

$E = \{(x,y,z) \mid 0 \le x \le 1,\ x^2 \le y \le x,\ 0 \le z \le x\}$

To set up integration in the order

$dy\,dx\,dz$

we have to:

- find all possible values for z
- given z, find all possible values for x
- given z and x, find all possible values for y.

Since $0 \le z \le x$ and x can be any number between 0 and 1, $0 \le x \le 1$, we see that z can attain any value between 0 and 1

$$0 \le z \le 1.$$

Now the inequalities $0 \le z \le x$ and $0 \le x \le 1$ show that if z is given, x can attain any value such that

$$z \le x \le 1.$$

Finally, if z and x are fixed, y will satisfy

$$x^2 \le y \le x$$

so

$$E = \{(x,y,z) \mid 0 \le z \le 1,\ z \le x \le 1,\ x^2 \le y \le x\}$$

and hence

$$\iiint_E (x+2y)\,dV = \int_0^1 \int_z^1 \int_{x^2}^{x} (x+2y)\,dy\,dx\,dz =$$

$$\int_0^1 \int_z^1 xy + y^2 \Big|_{y=x^2}^{y=x} dx\,dz =$$

$$\int_0^1 \int_z^1 2x^2 - (x^3 + x^4)\,dx\,dz =$$

$$\int_0^1 \frac{2}{3}x^3 - \frac{x^4}{4} - \frac{x^5}{5}\Big|_{x=z}^{x=1} dz =$$

$$\int_0^1 \left[\left(\frac{2}{3} - \frac{1}{4} - \frac{1}{5}\right) - \left(\frac{2}{3}z^3 - \frac{z^4}{4} - \frac{z^5}{5}\right)\right] dz =$$

$$\frac{13}{60} z - \frac{z^4}{6} + \frac{z^5}{20} + \frac{z^6}{30} \Big|_0^1 =$$

$$\frac{13}{60} - \frac{1}{6} + \frac{1}{20} + \frac{1}{30} = \frac{2}{15}$$

which is the same answer as in the previous example.

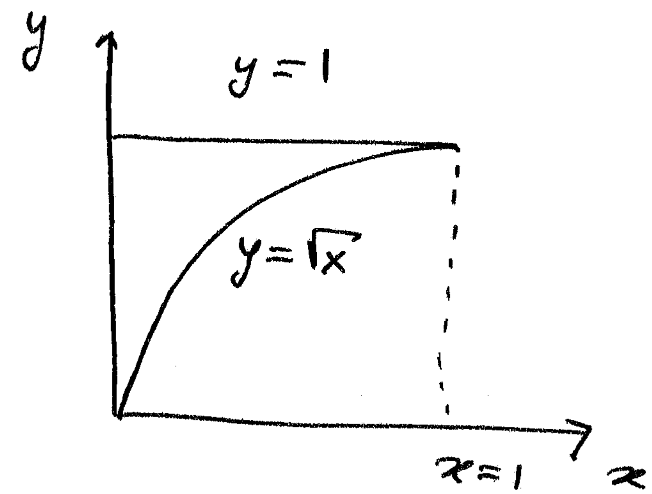
Example Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx \qquad \text{as}$$

$$\int_?^? \int_?^? \int_?^? f(x,y,z)\, dx\, dz\, dy$$

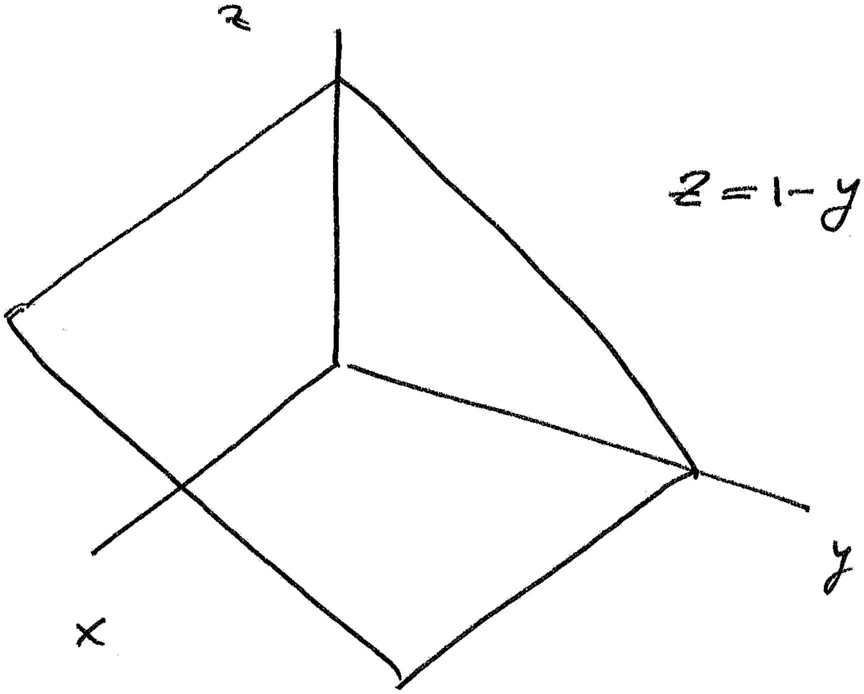
Solution First we find the shape of the domain of integration

$$E = \{ (x,y,z) \,/\, 0 \le x \le 1,\ \sqrt{x} \le y \le 1,\ 0 \le z \le 1-y \}$$

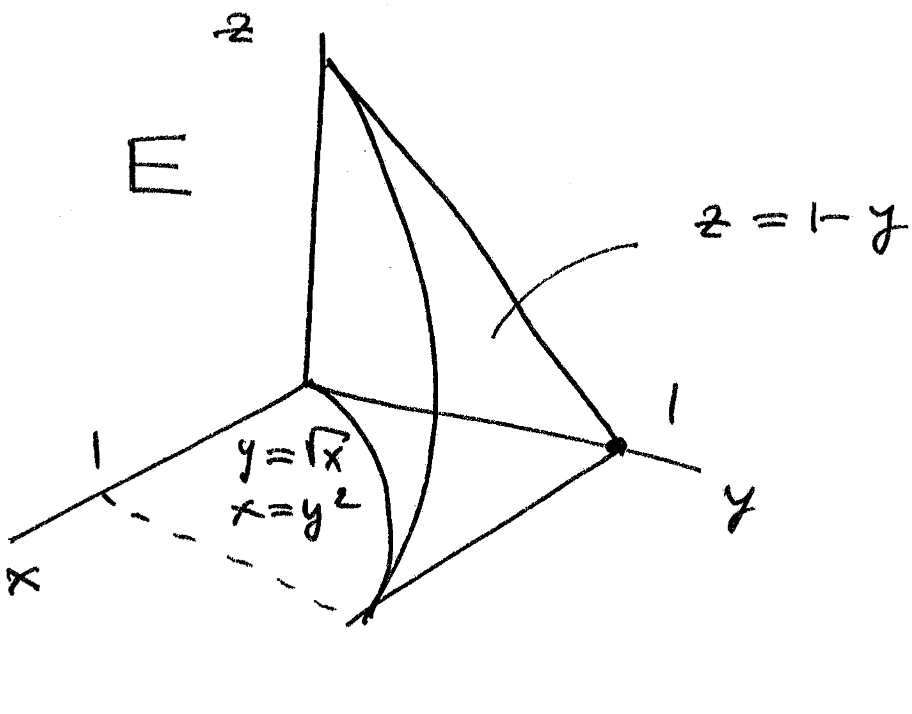


$0 \le x \le 1$

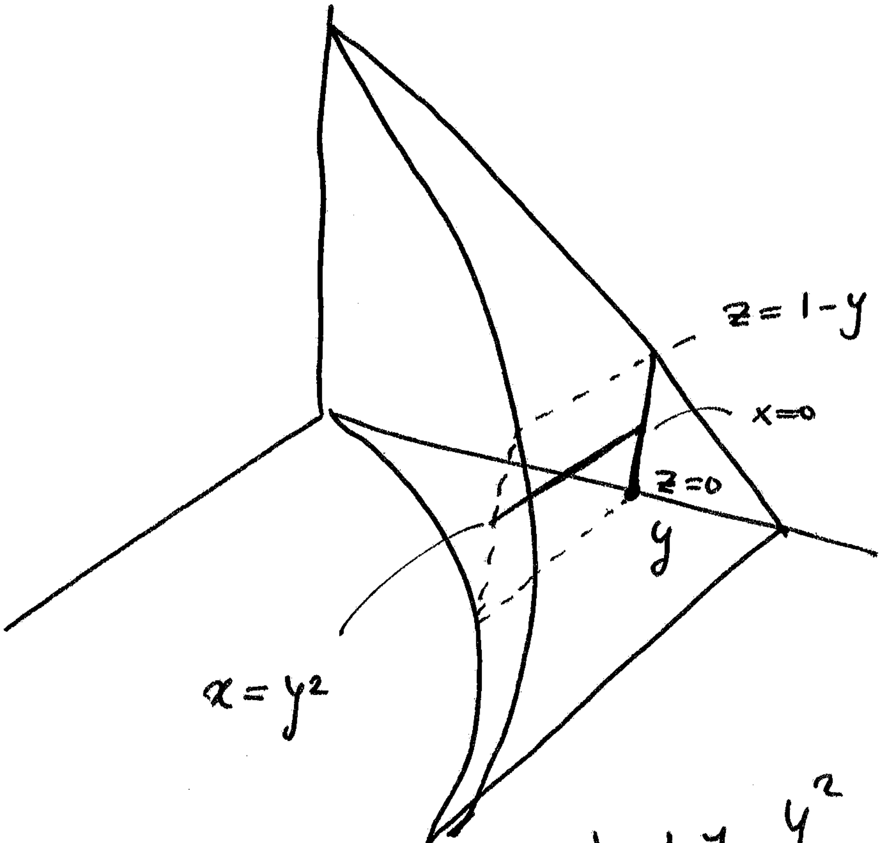
$\sqrt{x} \le y \le 1$



$z = 1 - y$ (plane)



Now we can describe E as follows



$0 \le y \le 1$

$0 \le z \le 1-y$

$0 \le x \le y^2$

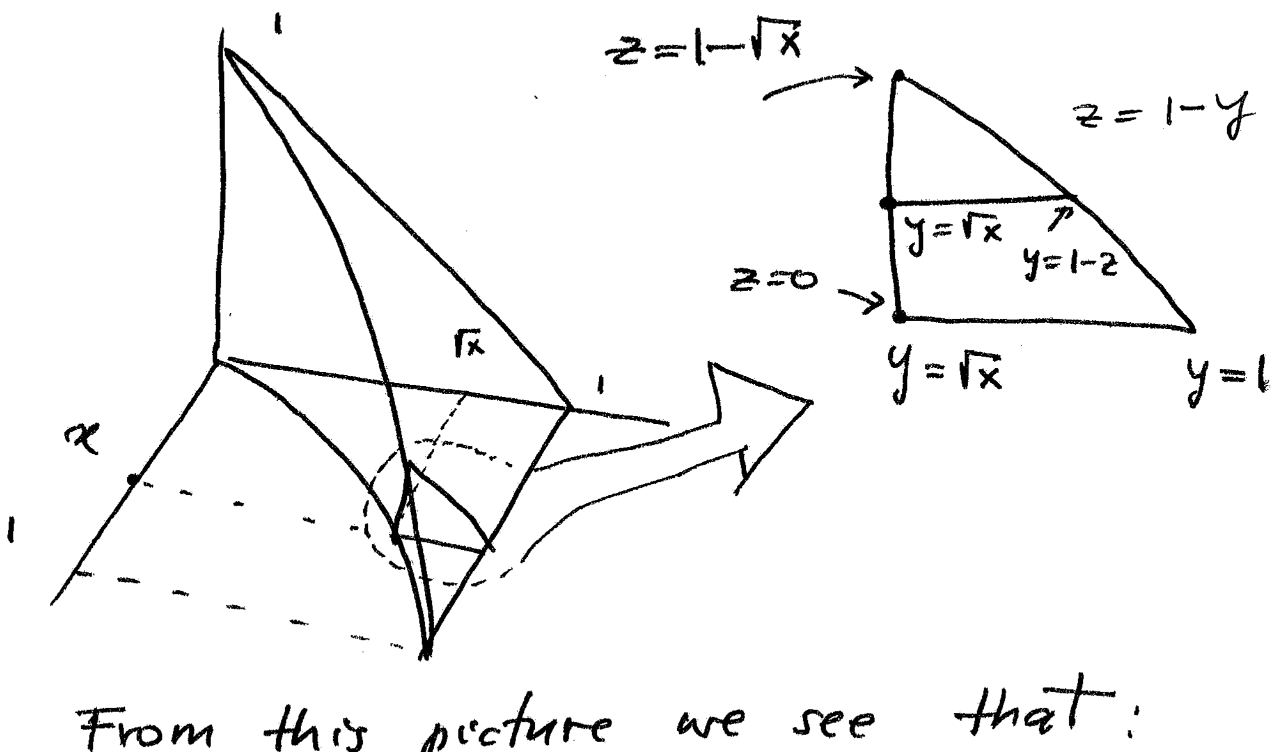
So

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

Example Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx \quad \text{as} \quad \int \int \int_?^? f(x,y,z)\,dy\,dz\,dx$$

This is the same integral as in the previous example so we know the shape of the domain. We need to find bounds for x, z, y in that order. Clearly $0 \leq x \leq 1$. Now we fix x and we need to find all possible values for z. Given $0 \leq x \leq 1$ the corresponding triangular section of the solid is shown below



From this picture we see that:
when we fix $0 \leq x \leq 1$, then

$$0 \leq z \leq 1 - \sqrt{x}$$

When we fix x and z, then

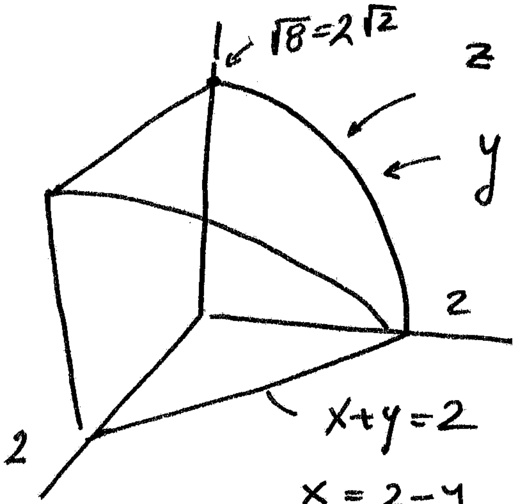
$$\sqrt{x} \leq y \leq 1 - z$$

Hence

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx.$$

<u>Example</u> Let E be the region in the first octant bounded by the surfaces $2y^2 + z^2 = 8$ and $x + y = 2$, and let $f(x,y,z)$ be a function whose domain contains E.

ⓐ Set up the integral over E as $\iiint_{?}^{?} f(x,y,z)\, dz\, dx\, dy$



$z = \sqrt{8 - 2y^2}$

$y = \sqrt{\dfrac{8 - z^2}{2}}$

$$\int_0^2 \int_0^{2-y} \int_0^{\sqrt{8-2y^2}} f(x,y,z)\, dz\, dx\, dy$$

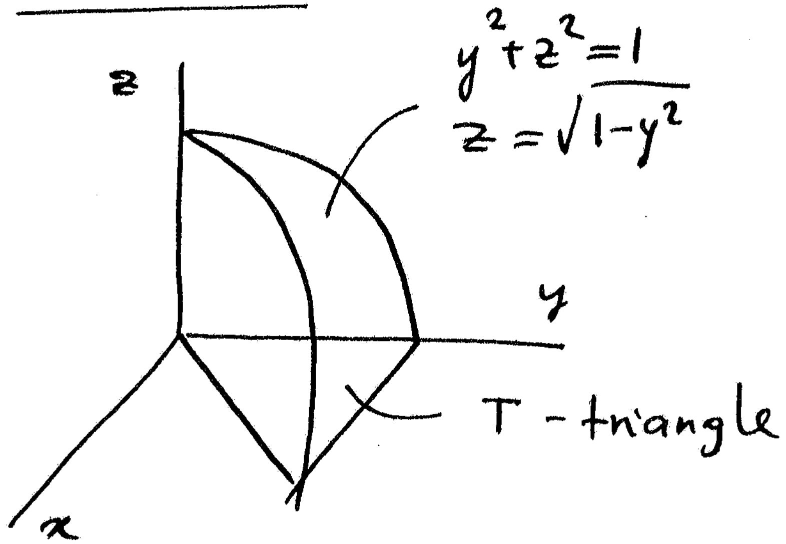
ⓑ Set up the integral over E as $\iiint_{?}^{?} f(x,y,z)\, dx\, dy\, dz$

$$\int_0^{\sqrt{8}} \int_0^{\sqrt{\frac{8-z^2}{2}}} \int_0^{2-y} f(x,y,z)\, dx\, dy\, dz$$

<u>Example</u> Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2+z^2 \leq 1$ by the planes $y = x$ and $x=0$. Evaluate the integral

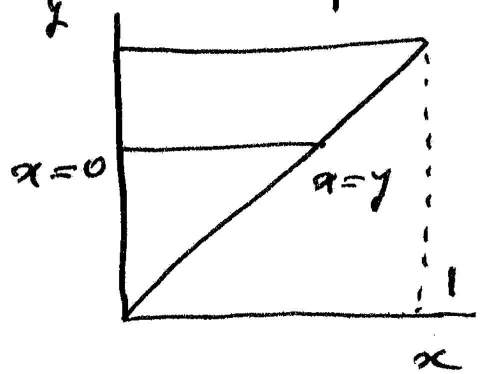
$$\iiint_G z \, dV$$

<u>Solution #1</u>



$$G = \{ (x, y, z) \mid (x,y) \in T, \; 0 \leq z \leq \sqrt{1-y^2} \}$$

T



$$T = \{ (x,y) \mid 0 \leq y \leq 1, \; 0 \leq x \leq y \}$$

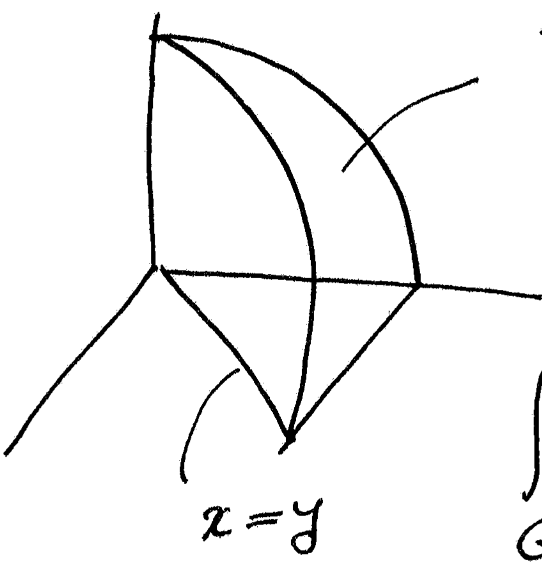
$$\iiint_G z\,dV = \iint_T \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy =$$

$$\iint_T \frac{z^2}{2}\Big|_{z=0}^{z=\sqrt{1-y^2}} dx\,dy = \iint_T \frac{1-y^2}{2}\,dx\,dy =$$

$$\int_0^1 \int_0^y \frac{1-y^2}{2}\,dx\,dy = \int_0^1 x\,\frac{1-y^2}{2}\Big|_{x=0}^{x=y} dy =$$

$$\int_0^1 y\,\frac{1-y^2}{2}\,dy = \int_0^1 \frac{y-y^3}{2}\,dy = \frac{y^2}{4} - \frac{y^4}{8}\Big|_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

Solution #2



$$D = \{(y,z) \mid y^2+z^2 \le 1,\ y,z \ge 0\}$$

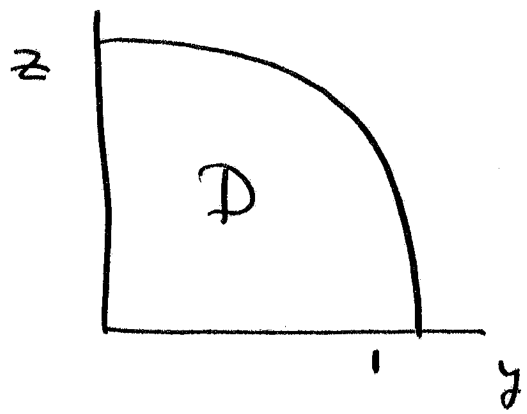
$$G = \{(x,y,z) \mid (y,z) \in D,\ 0 \le x \le y\}$$

$$\iiint_G z\,dV = \iint_D \int_0^y z\,dx\,dy\,dz =$$

$$\iint_D xz \Big|_{x=0}^{x=y} dy\,dz = \iint_D yz\,dy\,dz .$$

We can evaluate this integral using two different methods.

Method #1



$$y^2 + z^2 = 1$$

$$z = \sqrt{1-y^2}$$

$$0 \le y \le 1$$

$$0 \le z \le \sqrt{1-y^2}$$

$$\iint_D yz\,dy\,dz = \int_0^1 \int_0^{\sqrt{1-y^2}} yz\,dz\,dy =$$

$$\int_0^1 \frac{yz^2}{2}\Big|_{z=0}^{z=\sqrt{1-y^2}} dy = \int_0^1 \frac{y(1-y^2)}{2}\,dy =$$

$$\int_0^1 \frac{y}{2} - \frac{y^3}{2}\,dy = \frac{y^2}{4} - \frac{y^4}{8}\Big|_0^1 = \frac{1}{8} .$$

Method 2 We can use polar coordinates

$$D = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

These are polar coordinates in the yz - plane

$$y = r\cos\theta, \quad z = r\sin\theta, \quad dy\,dz = r\,dr\,d\theta$$

$$\iint_D yz\,dy\,dz = \int_0^{\pi/2}\int_0^1 r\cos\theta\, r\sin\theta\; r\,dr\,d\theta =$$

$$\int_0^{\pi/2} \cos\theta\sin\theta \int_0^1 r^3\,dr = \heartsuit$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos\theta\sin\theta = \frac{\sin(2\theta)}{2}$$

$$\heartsuit = \int_0^{\pi/2} \frac{\sin(2\theta)}{2}\,d\theta \int_0^1 r^3\,dr = \left.\frac{-\cos(2\theta)}{4}\right|_0^{\pi/2} \left.\frac{r^4}{4}\right|_0^1$$

$$= \frac{1}{8}.$$

Remark Let us look at Solution 2, Method 2 again. The key steps in our calculation look as follows

$$\iiint\limits_G z\,dV = \iint\limits_D \int_0^y z\,dx\,dy\,dz =$$

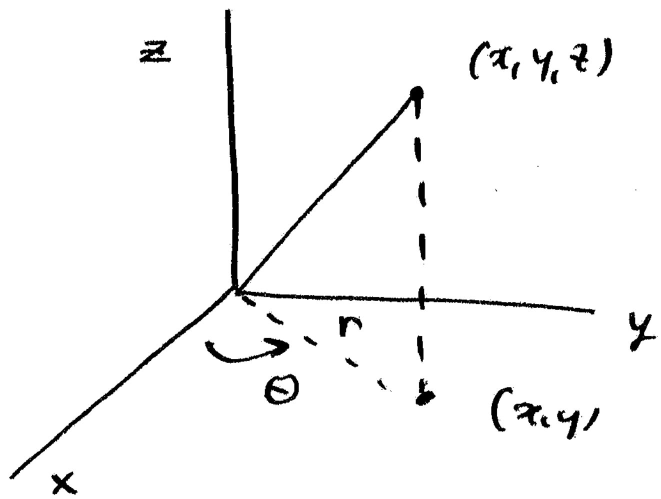
$$\iint\limits_D xz\Big|_{x=0}^{x=y} dy\,dz = \iint\limits_D yz\,dy\,dz =$$

$$\int_0^{\pi/2}\int_0^1 r\cos\theta\; r\sin\theta\; r\,dr\,d\theta = \frac{1}{8}.$$

Here we simply applied polar coordinates to variables y and z without making any change to variable x.

A method of integrating in 3D by applying polar coordinates to two of the three variables and leaving the third variable unchanged is called integration in cylindrical coordinates.

Cylindrical coordinates

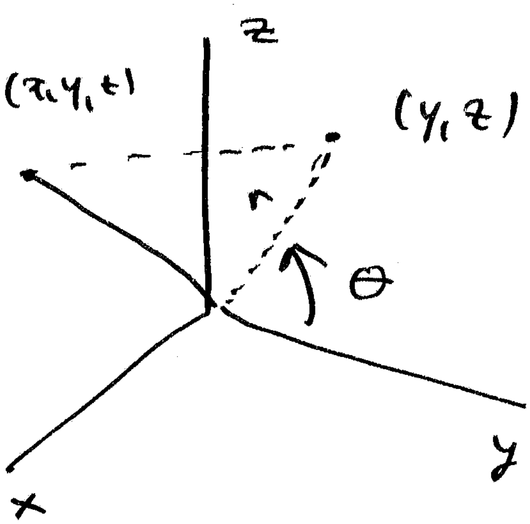


(r, θ, z) - cylindrical coordinates of the point (x, y, z)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

In other words we represent x and y in polar coordinates and leave z unchanged.

We can also use cylindrical coordinates with respect to other pair of variables



$$x = x$$
$$y = r\cos\theta$$
$$z = r\sin\theta$$

Such cylindrical coordinates were used on pp. 68–69.

We have already seen how to use cylindrical coordinates in the integration. Now we will describe it as a general method.

Suppose a solid E has the following description

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

where D has a convenient representation in polar coordinates

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA =$$

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta .$$

The formula seems complicated but it is actually easy. We just apply integration in polar coordinates to two of three variables, and we have already seen how it works.

Example Evaluate the integral

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx .$$

Solution The solid E over which we integrate has the following description

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{c} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \\ \sqrt{x^2+y^2} \le z \le 2 \end{array} \right\}$$

Observe that the conditions for x and y describe the disc of radius 2 which can be easily represented in polar coordinates. This suggests that we should evaluate the integral using cylindrical coordinates.

We have

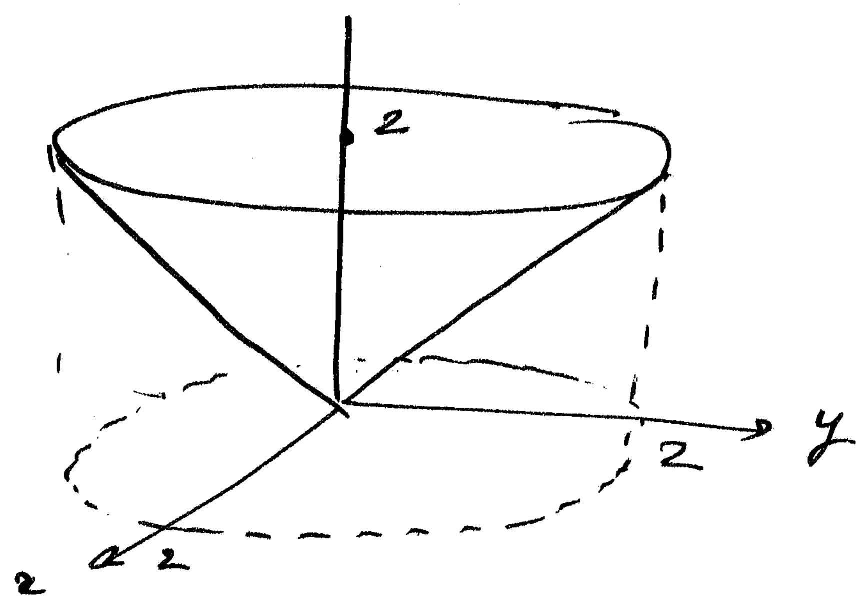
$$\iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_r^2 r^2\,dz\cdot r\,dr\,d\theta =$$

$$\int_0^{2\pi} d\theta \int_0^2 \int_r^2 r^3\,dz\,dr =$$

$$2\pi \int_0^2 z r^3\Big|_{z=r}^{z=2} dr = 2\pi \int_0^2 (2-r)r^3\,dr =$$

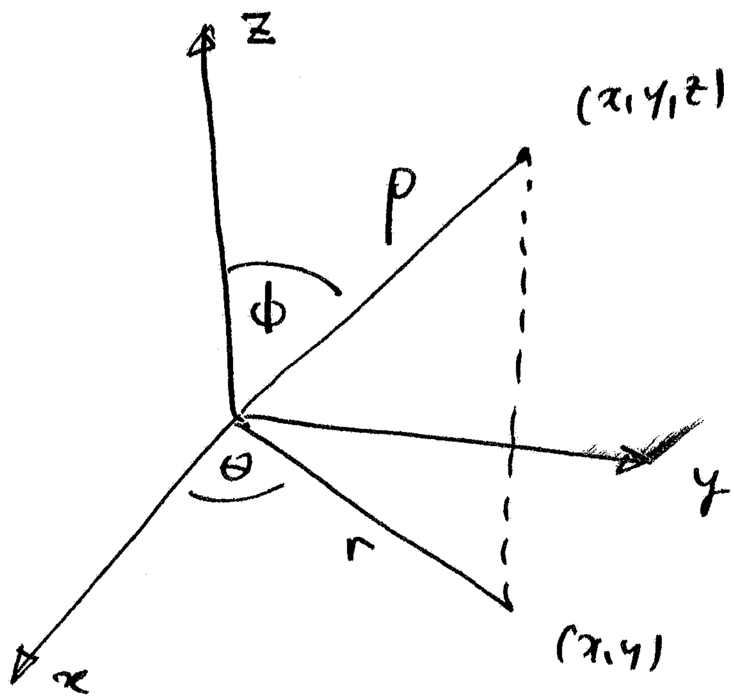
$$2\pi \int_0^2 2r^3 - r^4\,dr = 2\pi\left(\frac{r^4}{2} - \frac{r^5}{5}\right)\Big|_0^2 = \frac{16\pi}{5}.$$

We could solve the problem without sketching the solid E. It is actually a solid cone with the opening of 45°



Spherical coordinates

The cylindrical coordinates can hardly be be regarded as a good counterpart of the polar coordinates in the 3D space, because they are just polar coordinates apllied to two of the three variables. Nothing more than that. The coordinates that fully generalize the polar coordinates to the 3D space are so called <u>spherical coordinates</u> that we describe next.



Given a point (x,y,z) we denote by ρ the distance to the origin and by ϕ

the angle with the z - axis so $0 \leq \rho < \infty$, $0 \leq \phi \leq \pi$. Thus

$$z = \rho \cos \phi$$

The length r of the projection (x, y) on the xy plane equals $r = \rho \sin \phi$. Hence the polar coordinates of (x, y) are

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta.$$

This is to say that the spherical coordinates

(ρ, ϕ, θ), $0 \leq \rho < \infty$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$ and describe the position of (x, y, z) and

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \theta.$$

You know actually spherical coordinates from geography since ϕ is latitude and θ is longitude. The only difference is that in geography latitude is measured from the equator and we measure the

the angle ϕ from the North Pole.

Integration in spherical coordinates

When we express the integral in spherical coordinates we replace

x by $\rho \sin\phi \cos\theta$

y by $\rho \sin\phi \sin\theta$

z by $\rho \cos\theta$

and dV by $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

That means if a solid E can be expressed in spherical coordinates as

$$E = \{(\rho, \phi, \theta) \mid a \le \rho \le b, \ c \le \phi \le d, \ \alpha \le \theta \le \beta\}$$

then

$$\iiint_E f(x, y, z)\, dV =$$

$$\int_a^b \int_c^d \int_\alpha^\beta f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

Example Use spherical coordinates to find volume of the ball of radius R.

Solution The ball B of radius R has the following description in spherical coordinates

$$B = \{(\rho, \phi, \theta) \mid 0 \le \rho \le R,\ 0 \le \pi \le \pi,\ 0 \le \theta \le 2\pi\}$$

so

$$\text{Vol}(B) = \iiint_B dV = \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

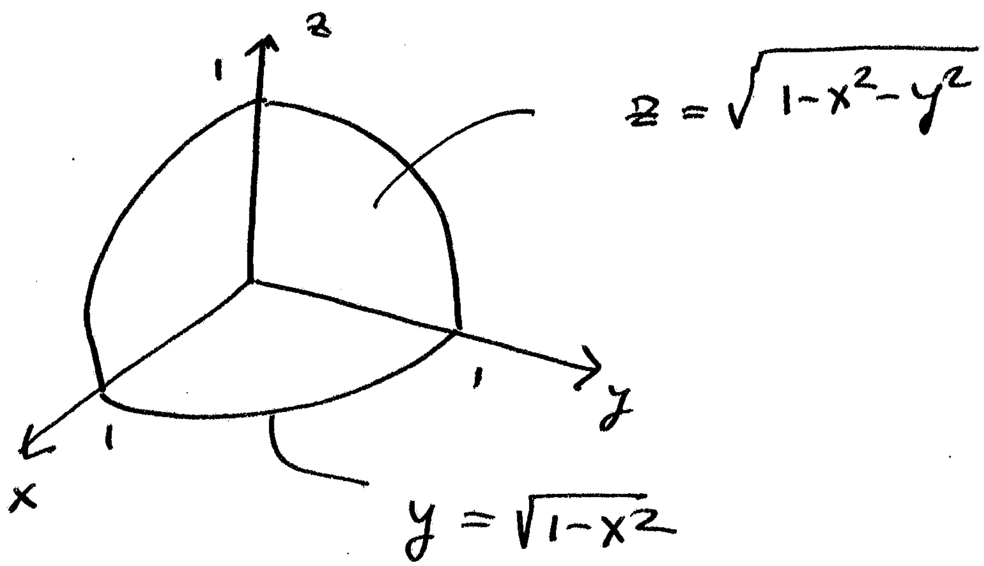
$$= \int_0^R \rho^2 d\rho \cdot \int_0^\pi \sin\phi \, d\phi \cdot \int_0^{2\pi} d\theta =$$

$$\frac{\rho^3}{3}\Big|_0^R \cdot (-\cos\phi)\Big|_0^\pi \cdot 2\pi = \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3.$$

Example Evaluate the integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \, dz \, dy \, dx.$$

Solution First we need to find the shape of the region over which



We integrate over a part of the unit ball that is contained in the first octant.

In spherical coordinates

$$E=\{(\rho,\phi,\theta)\,|\,0\le\theta\le\tfrac{\pi}{2},\ 0\le\phi\le\tfrac{\pi}{2},\ 0\le\rho\le 1\}$$

Hence

$$I=\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV=$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=$$

$$\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^1 e^{\rho^3}\rho^2\,d\rho$$

$$\frac{\pi}{2}\left[-\cos\phi\right]_0^{\pi/2}\ \frac{e^{\rho^3}}{3}\Bigg|_0^1=\frac{\pi}{2}\cdot 1\cdot\left(\frac{e}{3}-\frac{1}{3}\right)=\boxed{\frac{\pi(e-1)}{6}}$$

Example Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

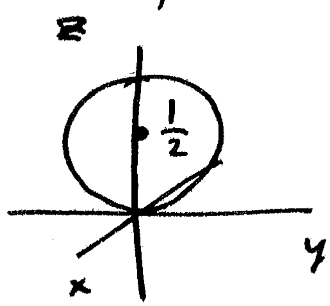
Solution First of all it is not obvious that the equation $x^2+y^2+z^2 = z$ represents a sphere but it does. Indeed,

$$x^2+y^2+z^2 = z$$

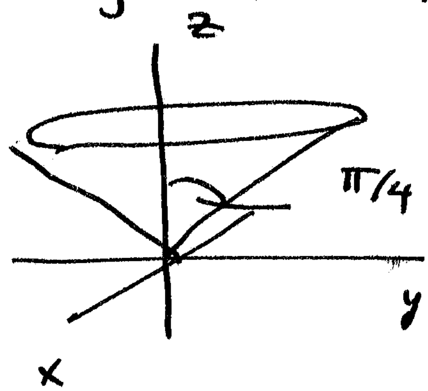
$$x^2+y^2+z^2 - 2\cdot\tfrac{1}{2}z + \tfrac{1}{4} = \tfrac{1}{4}$$

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

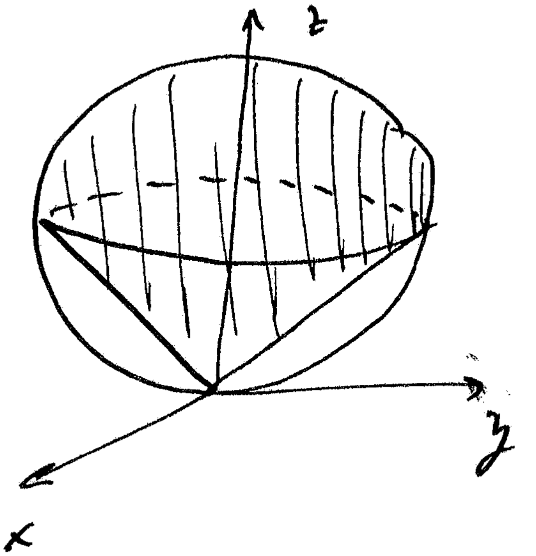
so it is the sphere of radius $\frac{1}{2}$ centered at the point $(0,0,\frac{1}{2})$ on the z-axis.



Now the cone $z = \sqrt{x^2+y^2}$ has the opening of $\pi/4$



Hence the solid above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$ has the shape of an ice cone. We want to represent this solid in spherical coordinates.



The sphere in the spherical coordinates has the representation

$$x^2+y^2+z^2 = z$$
$$\rho^2 = \rho \cos\phi$$
$$\rho = \cos\phi$$

Thus the interior of the sphere, the solid ball is given by

$$0 \leq \rho \leq \cos\phi, \qquad 0 \leq \phi \leq \pi/2$$

The cone has the opening of $\pi/4$ which is the angle ϕ with the z-axis so the equation of such cone is

$$\phi = \pi/4$$

Now the space above the cone is described by

$$0 \leq \phi \leq \pi/4.$$

Hence the solid above the cone and below the sphere is described by both equations so it is

$$E = \{ (\rho, \phi, \theta) \mid 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \cos\phi \}$$

Thus we compute volume of E as follows

$$\text{vol}(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left. \frac{\rho^3}{3} \right|_{\rho=0}^{\rho=\cos\phi} d\phi =$$

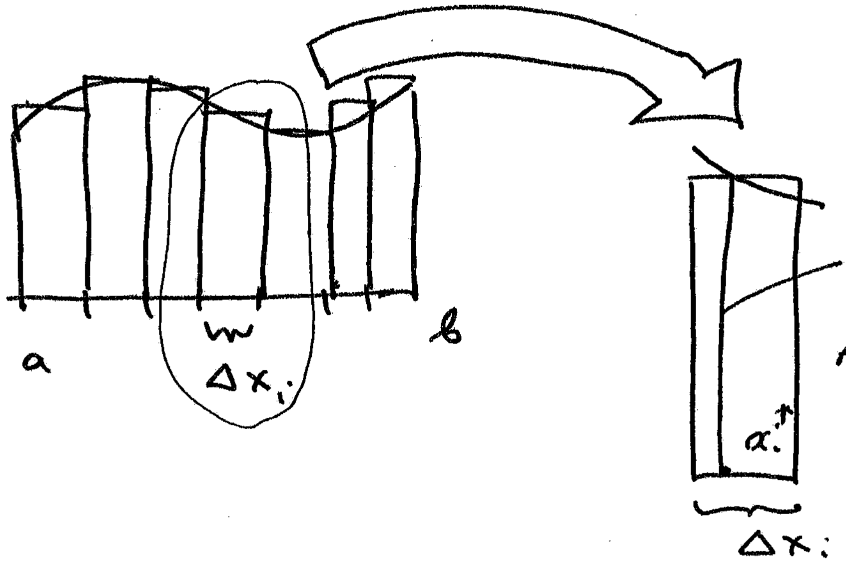
$$\frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \, (\cos\phi)^3 \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} =$$

$$\frac{2\pi}{3} \left[-\frac{(\sqrt{2}/2)^4}{4} + \frac{1}{4} \right] = \frac{\pi}{8}.$$

Line integrals

Let us recall that the integral $\int_a^b f(x)\,dx$ from Calculus I can be interpreted in terms of Riemann sums

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$



Area of the rectangle equals

$f(x_i^*)\,\Delta x_i$

The total area of all rectangles equals

$$\text{(RS)} \qquad \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

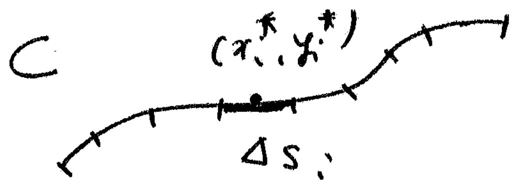
If the thickness Δx_i of the rectangles shrinks to 0 (and so $n \to \infty$), the Riemann sum (RS) will approach to

$$\int_a^b f(x)\,dx$$

In a similar way we want to define the integral

$$\int_C f(x,y)\,ds$$

where C is a planar curve and $f(x,y)$ is a continuous function defined at all points of C.



We partition the curve C into short pieces of lengths Δs_i, $i = 1, 2, \dots, n$. In each piece we select a point (x_i^*, y_i^*) and the corresponding Riemann sum is

$$\text{(RS)} \qquad \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

As the lengths Δs_i of pieces into which we partition the curve C tend to zero (and so the total number n of pieces tends to ∞), the Riemann sums (RS) will approach to

$$\int_C f(x,y)\,ds$$

so we can write

$$\int_C f(x,y)\,ds \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i .$$

<u>Interpretation</u> The integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the

graph of $y = f(x)$. A similar interpretation is available for the line integral

$$\int_C f(x,y)\,ds.$$

The curve C is in the xy plane and $z = f(x,y)$ is above the point $(x,y) \in C$. Thus we can think of a "kinked" graph of $z = f(x,y)$ above the curve C.

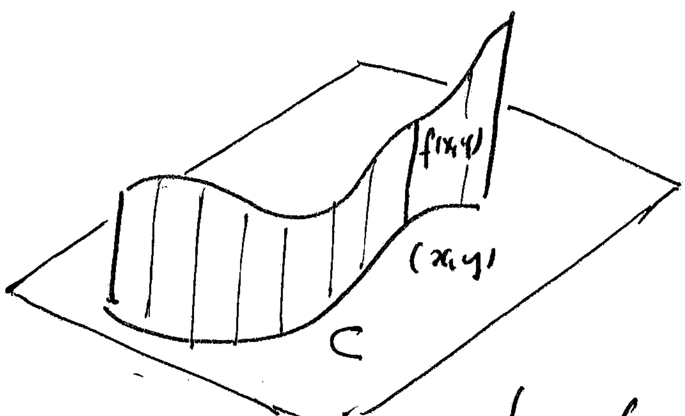


Fig 1.

The integral $\int_C f(x,y)\,ds$ represents the area of the surface under the kinked graph of $z = f(x,y)$.

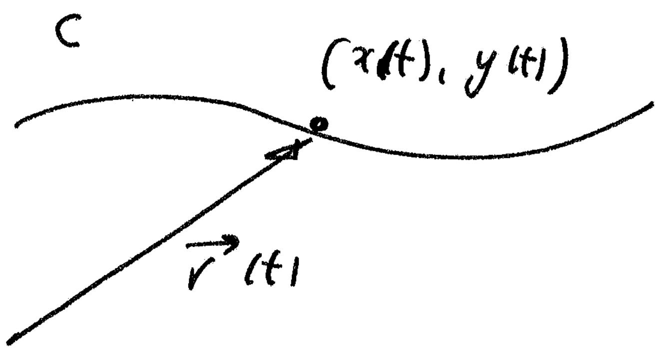
<u>Example</u> If $f = 1$ at all points, then

$$\int_C ds = \int_C 1\,ds = \text{length}(C).$$

This is clear. If the height $f(x,y)$ of the surface on Fig 1 is constant and equal 1, its area must be equal length (C).

How to evaluate the line integral

If $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$ is a parametrization of the curve C



then

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

Example

$$\int_C ds = \int_a^b \underset{\substack{\uparrow \\ f}}{1} \cdot \underbrace{|\vec{r}\,'(t)|}_{\text{speed}}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt = \text{length}(C).$$

This is consistent with what we already discussed on p. 84.

Example Evaluate the integral $\int_C 2 + x^2 y\,ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution We can parametrize C by
$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi$

so $x(t) = \cos t, \quad y(t) = \sin t$

and
$$|\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

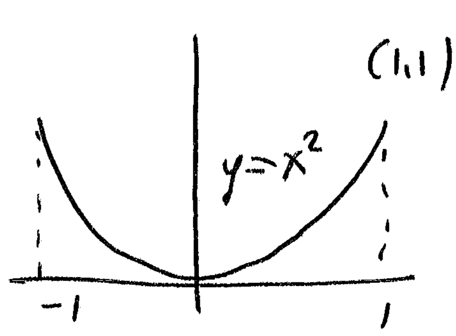
Thus
$$\int_C 2 + x^2 y \, ds = \int_0^{\pi} (2 + x(t)^2 y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t) \cdot 1 \, dt$$

$$= \left[2t - \frac{\cos^3 t}{3} \right]_0^{\pi} = 2\pi + \frac{2}{3}.$$

Example Evaluate the integral
$$\int_C y \sqrt{1 + 4x^2} \, ds$$
where C is the part of the parabola $y = x^2$ between the points $(-1,1)$ and $(1,1)$.

Solution



$\vec{r}(t) = \langle t, t^2 \rangle$
$-1 \le t \le 1$

$x(t) = t, \quad y(t) = t^2$

$$|\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1+4t^2}$$

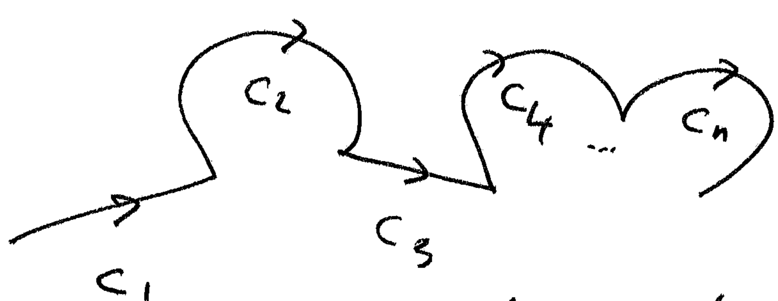
$$\int_C y\sqrt{1+4x^2}\,ds = \int_{-1}^{1} y(t)\sqrt{1+4x(t)^2}\,|\vec{r}\,'(t)|\,dt$$

$$= \int_{-1}^{1} t^2\sqrt{1+4t^2}\,\sqrt{1+4t^2}\,dt$$

$$= \int_{-1}^{1} t^2(1+4t^2)\,dt = \int_{-1}^{1} t^2+4t^4\,dt = \frac{t^3}{3}+\frac{4t^5}{5}\Big|_{-1}^{1} = \frac{2}{3}+\frac{8}{5}.$$

If a curve C is piecewise smooth and consists of smooth pieces $C_1, C_2, \ldots, C_n$ then

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds$$

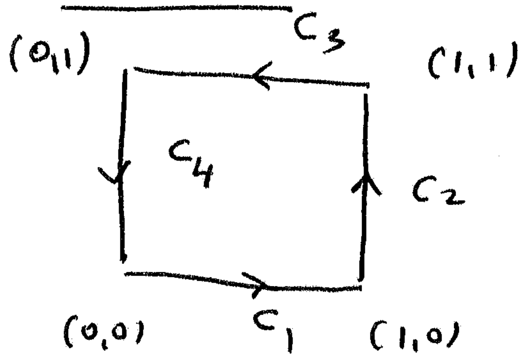


We compute each of the integrals on the right hand side separately and add them up.

Example Evaluate the integral

$\int_C xy\,ds$ where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(1,1)$.

Solution



$$C = C_1 + C_2 + C_3 + C_4$$

$$\int_C xy\,ds = \int_{C_1} xy\,ds + \int_{C_2} xy\,ds + \int_{C_3} xy\,ds + \int_{C_4} xy\,ds$$

On C_1, $y = 0$ so $\int_{C_1} xy\,ds = 0$

On C_4 $x = 0$ so $\int_{C_4} xy\,ds = 0$.

Thus

$$\int_C xy\,ds = \int_{C_2} xy\,ds + \int_{C_3} xy\,ds$$

C_2 : $\vec{r}(t) = \langle 1, t \rangle$, $0 \le t \le 1$

$x(t) = 1$, $y(t) = t$, $|\vec{r}\,'(t)| = 1$

$$\int_{C_2} xy\,ds = \int_0^1 1 \cdot t \cdot 1\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$C_3: \quad \vec{r}(t) = \langle 1-t, 1 \rangle, \quad 0 \le t \le 1$

$x(t) = 1-t, \quad y(t) = 1, \quad |\vec{r}\,'(t)| = 1$

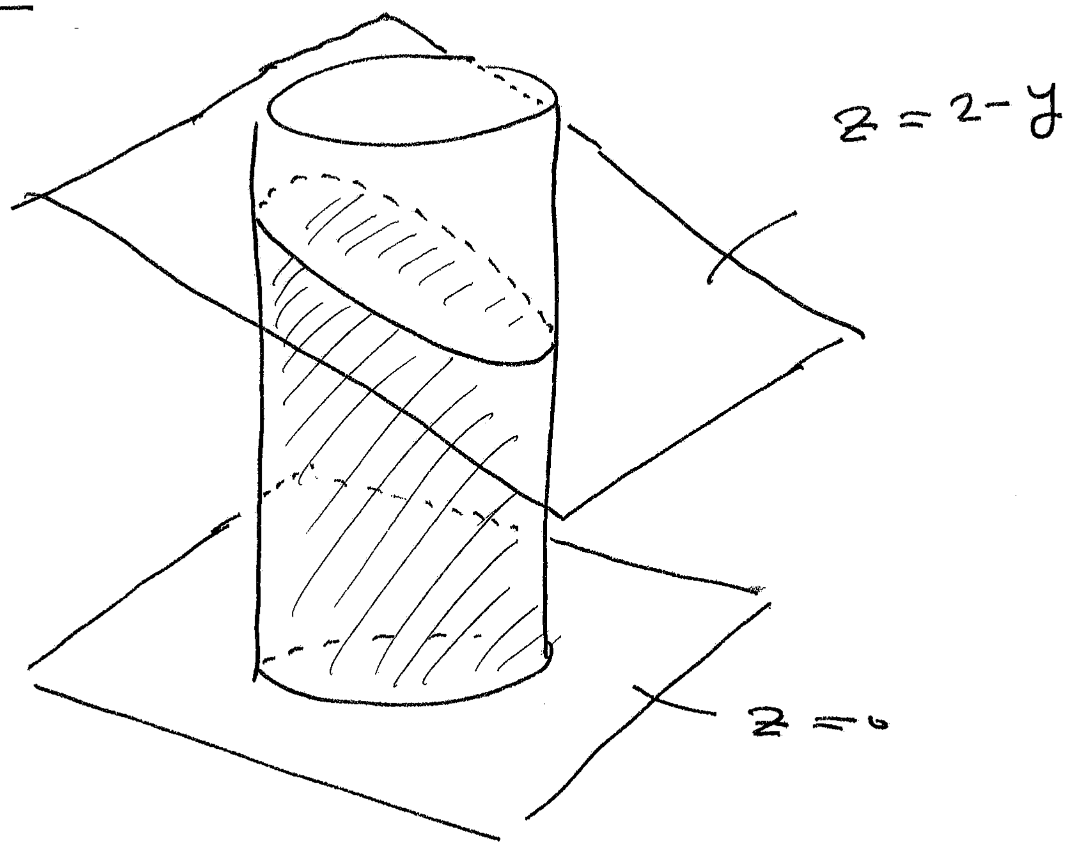
$$\int_{C_3} xy\,ds = \int_0^1 (1-t)\cdot 1 \cdot 1\, ds = \frac{1}{2}.$$

Hence

$$\int_C xy\,ds = \int_{C_2} xy\,ds + \int_{C_3} xy\,ds = \frac{1}{2} + \frac{1}{2} = 1.$$

<u>Example</u> Find the surface area of the part of the cylinder $x^2+y^2=1$ between the planes $z=0$ and $y+z=2$.

<u>Solution</u>



The surface whose area we need to find is the "kinked" graph of $f(x,y) = 2-y$ defined at the points of the circle C : $x^2+y^2=1$. Thus according to the discussion on p. 84,

$$\text{Area} = \int_C 2-y \, ds .$$

We can parametrize the circle C by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$ so $x(t) = \cos t$, $y(t) = \sin t$, $|\vec{r}\,'(t)| = 1$

$$\text{Area} = \int_0^{2\pi} (2-\sin t)\cdot 1 \, dt = 4\pi .$$

<u>Line integrals in space</u>

Similarly, if C is a curve in 3D space and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$ is its parametrization then the integral of $f(x,y,z)$ along C is defined by

$$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt .$$

Example Calculate $\int_C (x+y+z)\,ds$, where C is the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le \pi$.

Solution $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$, $|\vec{r}\,'(t)| = \sqrt{2}$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{2}\,dt$$

$$f(\vec{r}(t)) = f(x(t), y(t), z(t)) = x(t) + y(t) + z(t)$$
$$= \cos t + \sin t + t$$

$$f(\vec{r}(t))\,|\vec{r}\,'(t)| = (\cos t + \sin t + t)\sqrt{2}$$

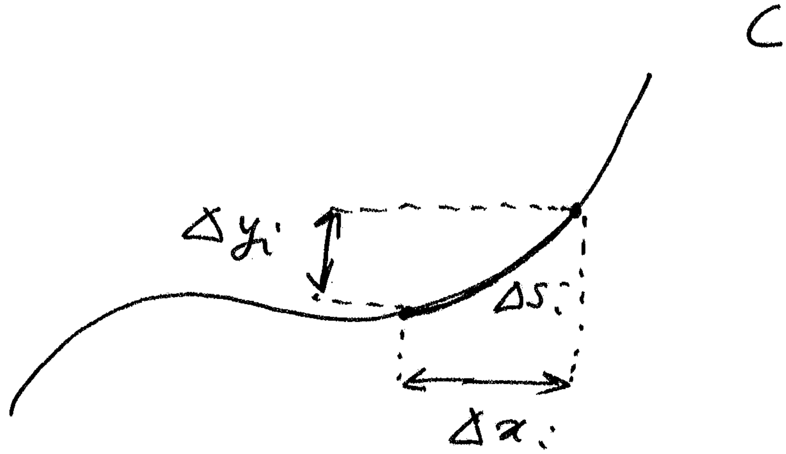
$$\int_C x+y+z\,ds = \int_0^{\pi} (\cos t + \sin t + t)\sqrt{2}\,dt = 2\sqrt{2} + \frac{\sqrt{2}}{2}\pi^2.$$

Line integrals of vector fields

Recall that

$$\int_C f(x,y)\,ds \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i.$$

Here Δs_i represents the increase of length of the curve. We can define similar integrals where Δs_i is replaced by Δx_i and Δy_i, the increase of the x coordinate and the y coordinate along the curve C



So

$$\int_C f(x,y)\,dx \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$

$$\int_C f(x,y)\,dy \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

We compute these integrals as follows. Given a parametrization $\vec{r}(t)$, $a \le t \le b$ of the curve C

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

Compare these formulas with

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

If we have two functions $P(x,y)$ and $Q(x,y)$ then we use the following notation that combines two different integrals

$$\int_C P(x,y)\,dx + Q(x,y)\,dy =$$

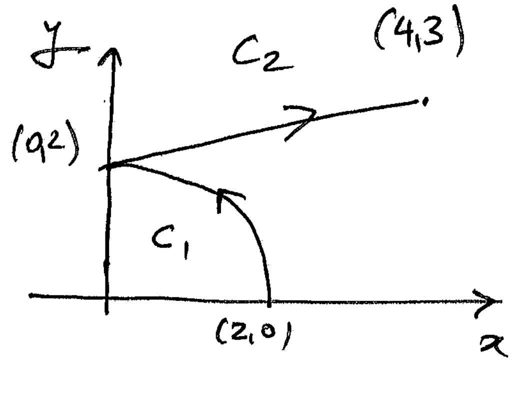
$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy =$$

$$\int_a^b P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\,dt.$$

<u>Example</u> Evaluate $\int_C x^2\,dx + y^2\,dy$ where C consists of the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.

<u>Solution</u>

$$\int_C x^2dx+y^2dy = \int_{C_1} x^2dx+y^2dy + \int_{C_2} x^2dx+y^2dy$$



C_1: $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle 2\cos t, 2\sin t \rangle$,
$0 \le t \le \pi/2$.

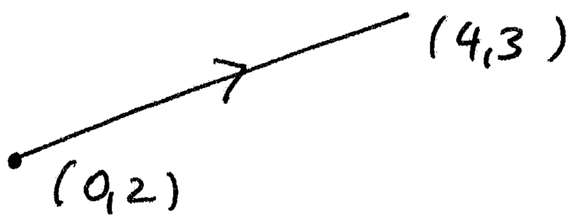
$$\int_{C_1} x^2 dx + y^2 dy = \int_0^{\pi/2} x(t)^2 x'(t) + y(t)^2 y'(t)\, dt$$

$$= \int_0^{\pi/2} (2\cos t)^2(-2\sin t) + (2\sin t)^2 (2\cos t)\, dt$$

$$= 8 \int_0^{\pi/2} -\cos^2 t \sin t + \sin^2 t \cos t \, dt$$

$$= 8 \left[\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \right]_0^{\pi/2} = 0$$

C_2:



$\vec{r}(t) = \langle 0, 2 \rangle (1-t) + \langle 4, 3 \rangle t$, $\quad 0 \le t \le 1$

$= \langle 0 + 4t, 2(1-t) + 3t \rangle = \langle 4t, 2+t \rangle$

$= \langle x(t), y(t) \rangle$

$$\int_{C_2} x^2 dx + y^2 dy = \int_0^1 x(t)^2 x'(t) + y(t)^2 y'(t)\, dt$$

$$= \int_0^1 (4t)^2 \cdot 4 + (2+t)^2 \cdot 1\, dt = 64\frac{t^3}{3} + \frac{(2+t)^3}{3}\Big|_0^1 = \frac{83}{3}$$

Thus

$$\int_C x^2dx+y^2dy = \int_{C_1} x^2dx+y^2dy + \int_{C_2} x^2dx+y^2dy = 0+\frac{83}{3} = \frac{83}{3}.$$

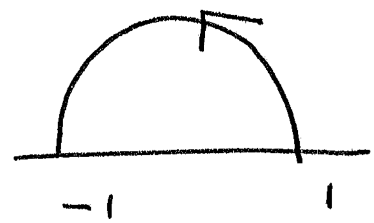
Remark Later we will see a different and a very quick method how to compute the integral from the above example.

The next example is very important as it shows an essential difference between the integrals

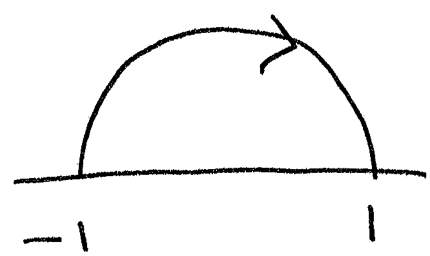
$$\int_C f\,dx,\quad \int_C f\,dy \quad \text{and} \quad \int_C f\,ds.$$

Example Consider two different parametrizations of the upper half of the unit circle

$\vec{r}_1(t) = \langle \cos t, \sin t\rangle,\ 0 \le t \le \pi$



$\vec{r}_2(t) = \langle -\cos t, \sin t\rangle,\ 0 \le t \le \pi$



Evaluate $\int_C y\,ds$ using both parametrizations $\vec{r}_1$ and $\vec{r}_2$

$\boxed{\vec{r}_1}$ $$\int_C y\,ds = \int_0^{\pi} \underbrace{\sin t}_{y(t)} \underbrace{|\vec{r}_1{}'(t)|}_{1}\,dt =$$

$$-\cos t \Big|_0^{\pi} = 2$$

$\boxed{\vec{r}_2}$ $$\int_C y\,ds = \int_0^{\pi} \underbrace{\sin t}_{y(t)} \underbrace{|\vec{r}_2{}'(t)|}_{1}\,dt =$$

$$-\cos t \Big|_0^{\pi} = 2.$$

The integral $\int_C y\,ds$ does not depend on the choice of a parametrization of C.

Evaluate $\int_C x^2\,dx$ using both parametrizations $\vec{r}_1$ and $\vec{r}_2$.

$\boxed{\vec{r}_1}$ $$\int_C x^2\,dx = \int_0^{\pi} x^2(t)\,x'(t)\,dt =$$

$$\int_0^{\pi} \cos^2 t(-\sin t)\,dt = \frac{\cos^3 t}{3}\Big|_0^{\pi} = -\frac{2}{3}$$

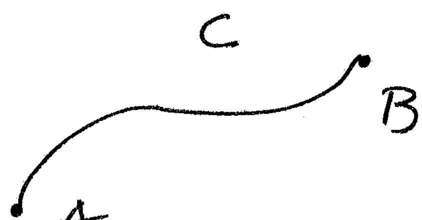
$\boxed{\vec{r}_2}$ $$\int_C x^2\,dx = \int_0^{\pi} x^2(t)\,x'(t)\,dt = \int_0^{\pi} (-\cos t)^2 \sin t\,dt$$

$$= \int_0^{\pi} \cos^2 t \sin t \, dt = \frac{\cos^3 t}{-3}\Big|_0^{\pi} = \frac{2}{3}.$$

Two different answers !!!

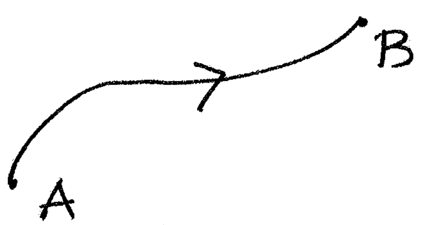
<u>In general</u>

Consider a geometric curve C that connects points A and B.



There are two essentially different ways to parametrize the curve C.

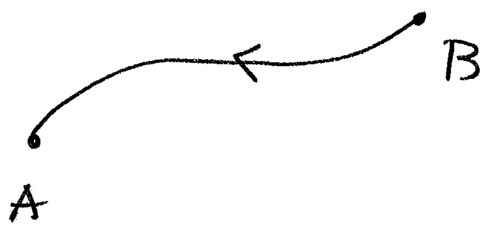
- One way of parametrization is to parametrize it from A to B. That means we consider a parametrization



$\vec{r}_1(t)$, $a \le t \le b$

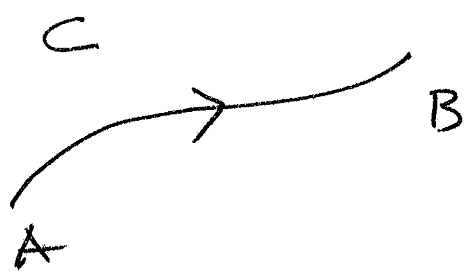
that starts at $\vec{r}_1(a) = A$ and ends at $\vec{r}_1(b) = B$. This is indicated by the direction of the arrow on the picture.

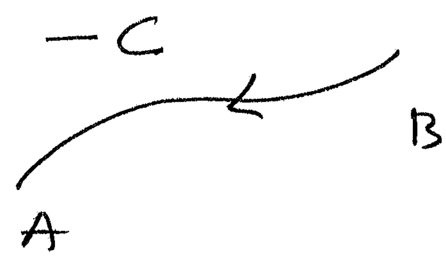
- Another way is to parametrize it from B to A. That means we consider a parametrization $\vec{r}_2(t)$, $a \le t \le b$



that starts at $\vec{r}_2(a) = B$ and ends at $\vec{r}_1(b) = A$. Again the direction of the parametrization is indicated by the arrow on the above picture.

If we denote the parametrization by $\vec{r}_1$ by C, we denote the parametrization by $\vec{r}_2$ by $-C$. The "$-$" sign indicates the change of orientation of the curve i.e. it indicates the reverse direction of the parametrization





In the example discussed on p. 96 we have seen that

$$\underbrace{\int_C y\,ds}_{\vec{r_1}\text{ param.}} = \underbrace{\int_{-C} y\,ds}_{\vec{r_2}\text{ param.}}$$

However

$$\underbrace{\int_C x^2\,dx}_{\vec{r_1}\text{ param.}} = -\underbrace{\int_{-C} x^2\,dx}_{\vec{r_2}\text{ param.}}$$

This is a general fact

$$\int_C f\,ds = \int_{-C} f\,ds$$

but

$$\int_C f\,dx = -\int_{-C} f\,dx$$

$$\int_C f\,dy = -\int_{-C} f\,dy.$$

More line integrals

Similarly as in $\mathbb{R}^2$ we can define the following line integrals in $\mathbb{R}^3$

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$$

Thus

$$\int_C P\,dx + Q\,dy + R\,dz =$$

$$\int_a^b P(x(t),y(t),z(t))\,x'(t) + Q(\dots)\,y'(t) + R(\dots)\,z'(t)\,dt.$$

Now we will define one more line integral.

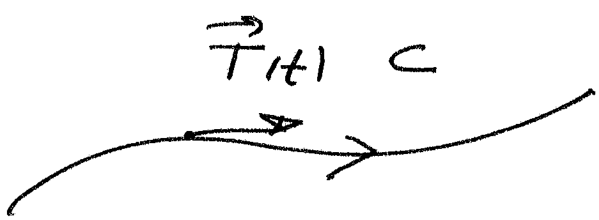
If $\vec{F} = \langle P, Q, R \rangle$ is a vector field, then we define

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

If $\vec{r}(t)$ is a parametrization of a curve C, then

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

is the unit tangent vector to the curve and the direction of the vector $\vec{T}$ is consistent with the orientation of C i.e. with this direction in which the curve C is parametrized. It is explained on the picture below



Since $\vec{F} \cdot \vec{T}$ is the number valued function, we can talk about the integral

$$\int_C \vec{F} \cdot \vec{T} \, ds$$

The next result shows how different types of the integrals are connected.

Theorem If $\vec{F} = \langle P, Q, R \rangle$ is a continuous vector field in $\mathbb{R}^3$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C P\,dx + Q\,dy + R\,dz$$

Proof ① $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

② $$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}}_{\vec{T}(t)} \cancel{|\vec{r}\,'(t)|}\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

③ $$\int_C P\,dx + Q\,dy + R\,dz =$$

$$= \int_C P(\underbrace{x(t), y(t), z(t)}_{\vec{r}(t)})\, x'(t) + Q(\dots)\, y'(t) + R(\dots)\, z'(t)\,dt =$$

$$= \int_C \underbrace{\langle P(\vec{r}(t)), Q(\vec{r}(t)), R(\vec{r}(t)) \rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle x'(t), y'(t), z'(t) \rangle}_{\vec{r}\,'(t)}\,dt$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

We proved that each of the integrals ①, ②, ③ from the theorem is equal to

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

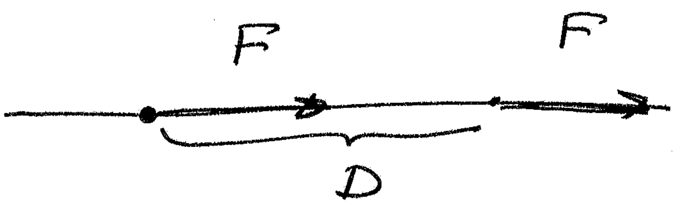
so the integrals ①, ② and ③ are equal which proves the theorem.

<u>Work</u> The line integral $\int_C \vec{F} \cdot d\vec{r}$ can be interpreted as work done by the force $\vec{F}$ along the curve C.

As we know

work = Force · displacement

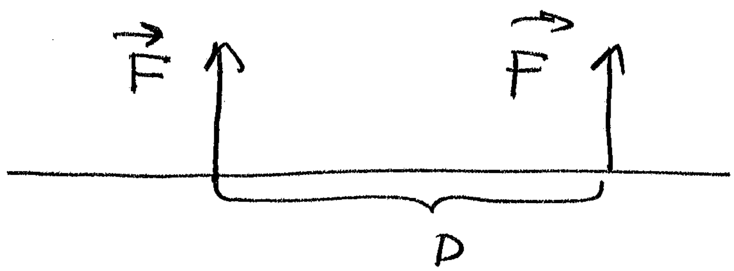
Ⓐ



$W = FD$

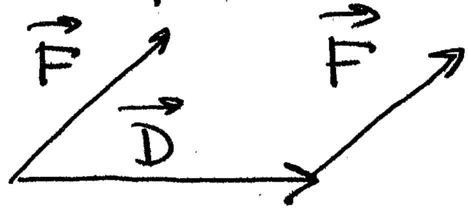
But if the displacement is in the direction orthogonal to force

Ⓑ



$W = 0$

It follows that if $\vec{F}$ is a constant force and $\vec{D}$ is the displacement vector, then the work equals

Ⓒ 

$$W = \vec{F} \cdot \vec{D}$$

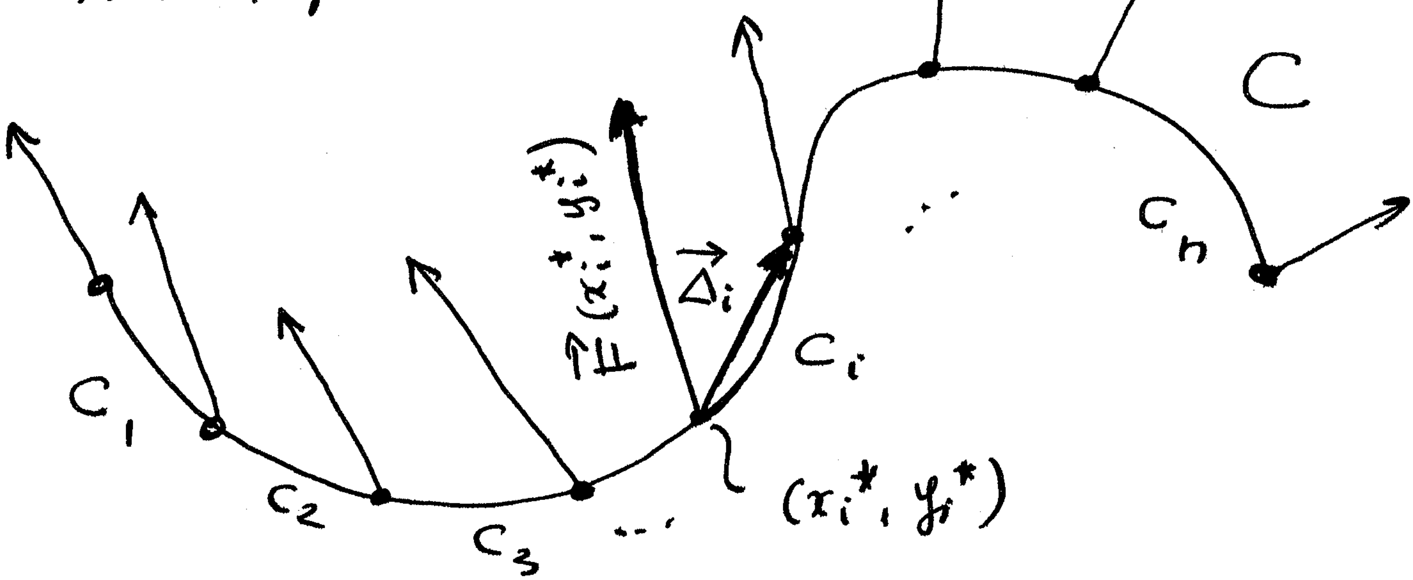
This formula includes both cases Ⓐ & Ⓑ and Ⓒ easily follows from Ⓐ & Ⓑ, because the vector $\vec{F}$ can be uniquely written as the sum of a vector Ⓐ parallel to $\vec{D}$ and Ⓑ orthogonal to $\vec{D}$

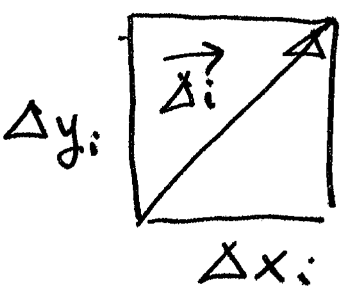
Suppose now that a force $\vec{F}$ is <u>not</u> constant and the displacement is <u>not</u> along a vector $\vec{D}$, but along a curve C that has a parametrization $\vec{r}(t)$, $a \le t \le b$. Then

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$

We will explain the formula in the 2D case, but the same reasoning works in the 3D case as well.

Thus, we assume that $\vec{F} = \langle P, Q \rangle$ is a vector field in the xy plane that represents force





$$\vec{\Delta}_i = \langle \Delta x_i, \Delta y_i \rangle$$

We partition the curve C into small pieces $C = C_1 + C_2 + \dots + C_n$. Each piece C_i can be approximated by the displacement vector $\vec{\Delta}_i$. Thus the work along C_i equals approximately

$$W_i \approx \vec{F}(x_i^*, y_i^*) \cdot \vec{\Delta} x_i =$$

$$\langle P(x_i^*, y_i^*), Q(x_i^*, y_i^*) \rangle \cdot \langle \Delta x_i, \Delta y_i \rangle =$$

$$P(x_i^*, y_i^*) \Delta x_i + Q(x_i^*, y_i^*) \Delta y_i$$

and hence total work equals

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} P(x_i^*, y_i^*) \Delta x_i + \sum_{i=1}^{n} Q(x_i^*, y_i^*) \Delta y_i$$

$$\approx \int_C P\,dx + \int_C Q\,dy = \int_C \vec{F} \cdot d\vec{r}$$

Theorem p. 102

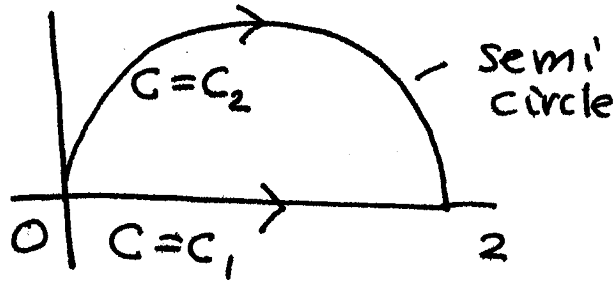
Riemann sum approximation of the integrals $\int_C P\,dx$ & $\int_C Q\,dy$.

Fundamental Theorem of the Line Integrals

Let us start with the following example

Example Evaluate the integral

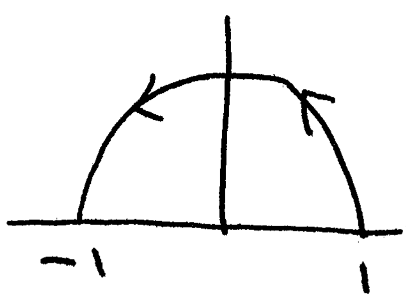
$\int_C x^2 dx + y^2 dy$, where



Solution $\boxed{C=C_1}$ $\vec{r}(t) = \langle t, 0\rangle$, $0 \le t \le 2$.

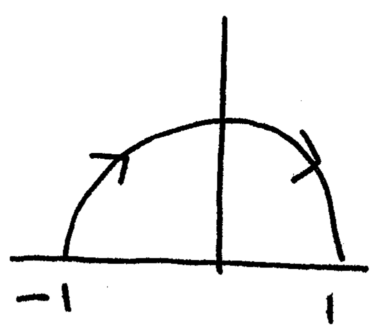
$$\int_C x^2 dx + y^2 dy = \int_0^2 t^2 \cdot 1 + 0\, dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3}.$$

$\boxed{C=C_2}$ First we parametrize $C=C_2$.



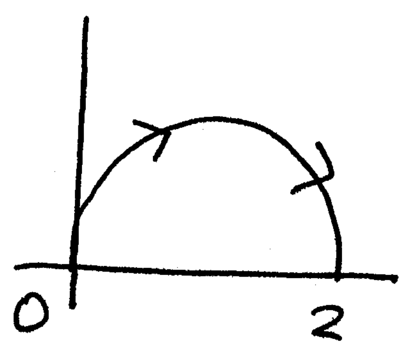
$\langle \cos t, \sin t\rangle$

$0 \le t \le \pi$



$\langle -\cos t, \sin t\rangle$

$0 \le t \le \pi$



$\langle 1-\cos t, \sin t\rangle$

$0 \le t \le \pi$

Thus $\vec{r}(t) = \langle 1-\cos t, \sin t \rangle$, $0 \le t \le \pi$

is a parametrization of the curve C_2.

Using this parametrization, the integral equals

$$\int_{C_2} x^2 dx + y^2 dy = \int_0^{\pi} x(t)^2 x'(t) + y(t)^2 y'(t)\, dt = \heartsuit$$

Now, we should substitute $x(t) = 1-\cos t$, $x'(t) = \sin t$, $y(t) = \sin t$, $y'(t) = \cos t$, but instead of that we will use a very nice trick

$$\heartsuit = \frac{1}{3} \int_0^{\pi} \left(x(t)^3 + y(t)^3\right)' dt =$$

$$\frac{1}{3}\left(x(t)^3 + y(t)^3\right)\Big|_0^{\pi} =$$

$$\frac{1}{3}\left((1-\cos t)^3 + (\sin t)^3\right)\Big|_0^{\pi} =$$

$$\frac{1}{3}\left[\left((1-(-1))^3 + 0^3\right) - \left((1-1)^3 + 0^3\right)\right] = \frac{8}{3}$$

$\square$

The computation of the integral along C_1 was easy and along C_2 was tricky, but in both cases we obtained the same answer

$$(*) \quad \int_{C_1} x^2 dx + y^2 dy = \int_{C_2} x^2 dx + y^2 dy = \frac{8}{3}.$$

In general, such integrals need not be equal. For example if the curves C_1 and C_2 are exactly the same as in the above example, then

$$\int_{C_1} y^2 dx + x^2 dy \neq \int_{C_2} y^2 dx + x^2 dy$$

(Check it!).

Is there any special reason why the integrals in (*) are equal, or it is a ~~spec~~ strange coincidence?

It turns out, it is not just a coincidence, but a consequence of the Fundamental Theorem of line integrals that we discuss next.

Recall that if f is a differentiable function of 2 or 3 variables, then

$\vec{F} := \nabla f = \langle f_x, f_y \rangle$ (or $\langle f_x, f_y, f_z \rangle$)

is a vector field so we can consider the line integrals:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

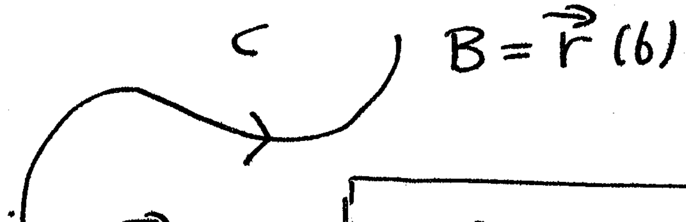
<u>Theorem (Fundamental theorem of line integrals)</u>

If $\vec{F} = \nabla f$ and C is a smooth curve parametrized by $\vec{r}(t)$, $a \le t \le b$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

If $\vec{F} = \nabla f$ for some function f, we say that the vector field $\vec{F}$ is <u>conservative</u> and we call the function f a <u>potential</u> of $\vec{F}$.

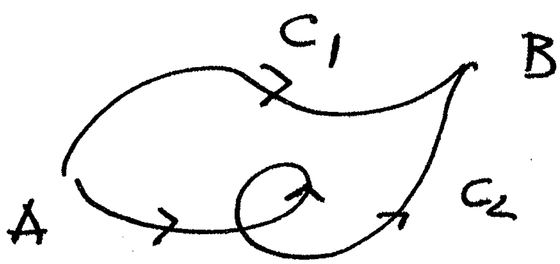
Note that the points $\vec{r}(b)$ and $\vec{r}(a)$ are the endpoints of the curve C.



$A = \vec{r}(a)$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)}$$

Hence, if we have two curves C_1, C_2 with the same endpoints



$$F = \nabla f$$

then
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(B) - f(A).$$

That means, if the vector field $\vec{F} = \nabla f$ is the gradient of a function, then the line integral $\int_C \vec{F}\cdot d\vec{r}$ does not depend on the shape of the curve C, but only on the location of the endpoints.

This is precisely the situation in the Example on p. 107. In that example we have to evaluate

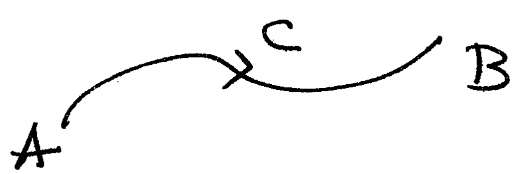
$$\int_C x^2dx + y^2dy = \int_C \vec{F}\cdot d\vec{r}, \text{ where } \vec{F} = \langle x^2, y^2 \rangle.$$

If $f(x,y) = \frac{1}{3}(x^3+y^3)$, then

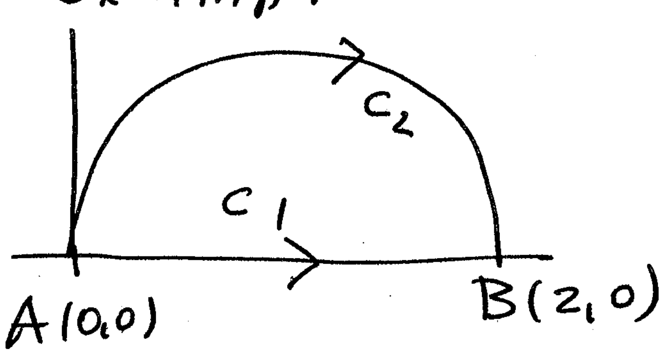
$$\nabla f = \langle f_x, f_y \rangle = \langle x^2, y^2 \rangle = \vec{F}$$

and hence

$$\int_C x^2dx + y^2dy = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



If the curves C_1 & C_2 are as in the example



then

$$\int_{C_1} x^2dx+y^2dy = \int_{C_2} x^2dx+y^2dy =$$

$$f(B)-f(A) = f(2,0)-f(0,0) =$$

$$\frac{1}{3}(2^3+0^3) - \frac{1}{3}(0^3+0^3) = \frac{8}{3}$$

which is the same answer as in (*) on p.109.

Proof of the Fundamental Theorem

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt =$$

$$\int_a^b \left\langle \frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t)), \frac{\partial f}{\partial z}(\vec{r}(t)) \right\rangle \cdot \left\langle \frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t) \right\rangle dt =$$

chain rule

$$\int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \overset{\downarrow}{=}$$

Fundamental theorem of calculus

$$\int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\,dt \overset{\downarrow}{=}$$

$$f(x(t), y(t), z(t)) \Big|_{t=a}^{t=b} =$$

$$f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)) =$$

$$f(\vec{r}(b)) - f(\vec{r}(a)) . \qquad \square$$

The following calculation will be needed in the next application

Example Find ∇f, where

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Solution $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ so

$$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -\frac{x}{(\sqrt{x^2+y^2+z^2})^3}$$

Similarly,

$$f_y = -\frac{y}{(\sqrt{x^2+y^2+z^2})^3}, \quad f_z = -\frac{z}{(\sqrt{x^2+y^2+z^2})^3}.$$

Hence

$$\nabla f = -\frac{\langle x, y, z\rangle}{(\sqrt{x^2+y^2+z^2})^3}.$$

If we write $\vec{X} = \langle x, y, z\rangle$, then

$$\nabla f(x,y,z) = -\frac{\vec{X}}{|\vec{X}|^3}.$$

Example Find the work done by the gravitational field

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$

in moving a particle with mass m from $(3,4,12)$ to $(2,2,0)$ along any smooth curve.

<u>Solution</u> The previous example shows that if

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}, \text{ then } \nabla f = \vec{F}.$$

Hence,

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} =$$

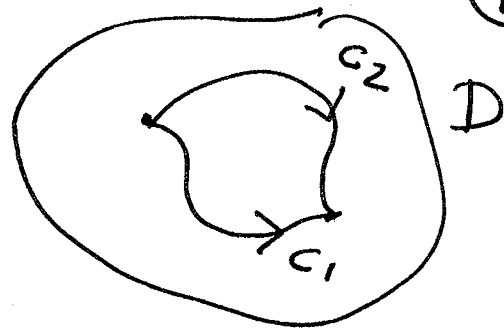
$$f(2,2,0) - f(3,4,12) =$$

$$\frac{mMG}{\sqrt{2^2+2^2+0^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right).$$

Path independence

Suppose that a vector field $\vec{F}$ is defined in a domain D. We say that the integral $\int_C \vec{F}\cdot d\vec{r}$ is <u>independent of path</u> if its value depends only on the location of the endpoints and not on the shape of the curve in D connecting the given endpoints. That is $\int_C \vec{F}\cdot d\vec{r}$ is <u>independent of path</u> if

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r},$$



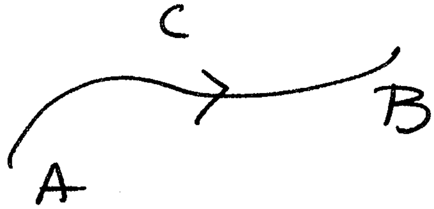
whenever the curves C_1 and C_2 in D have the same endpoints.

The fundamental theorem of line integrals has the following interpretation

If $\vec{F} = \nabla f$, then the integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path, that is, the integral of a conservative vector field is path independent.

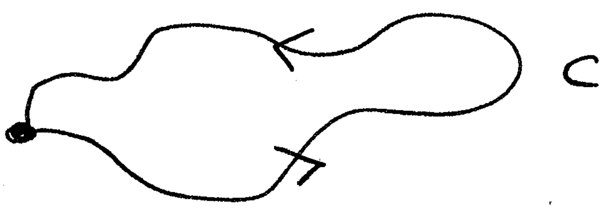
If we can find a function f such that $\vec{F} = \nabla f$, then we know that $\vec{F}$ is path independent.
However, if we are given a vector field $\vec{F}$ and we don't know f, how can we determine whether the integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path?
We will see now how to solve this problem.

Recall that the endpoints of a curve C are called



A - initial point
B - terminal point.

We say that a curve C is closed if the initial point is the same as the terminal point

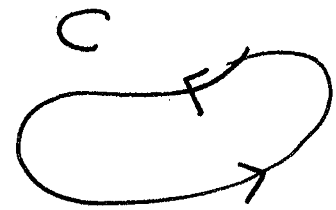


If $\vec{r}(t)$, $a \le t \le b$ is a parametrization of C, then C is closed if $\vec{r}(a) = \vec{r}(b)$.

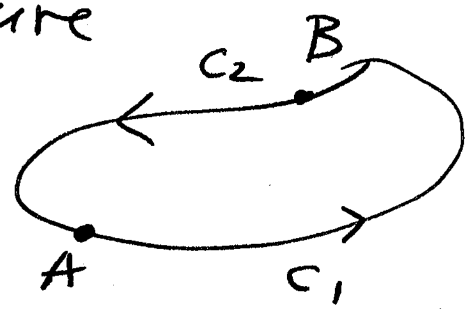
Theorem $\int_C \vec{F} \cdot d\vec{r}$ is path independent if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.

Proof. Suppose that $\int_C \vec{F} \cdot d\vec{r}$ is path independent and C is a closed curve. We need to show that

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

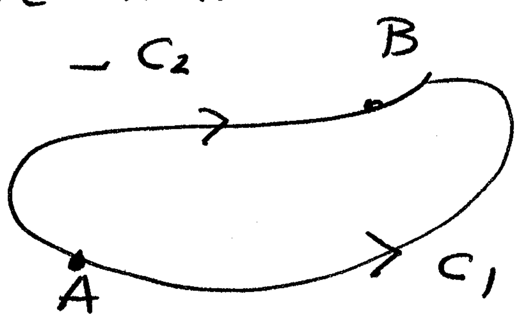


Choose two points A, B on C and we write $C = C_1 + C_2$ as in the picture



$C = C_1 + C_2$.

If we change orientation of C_2, then we have



The curves C_1 and $-C_2$ have the same endpoints A & B. Since the integral is path independent

$$(*) \quad \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-C_2} \vec{F}\cdot d\vec{r}$$

Therefore,

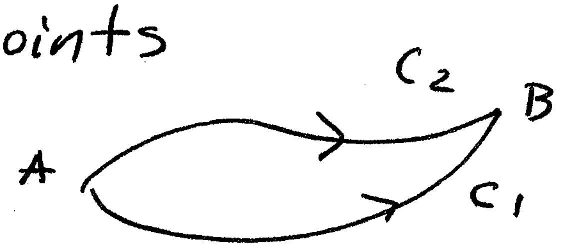
$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} =$$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} \overset{(*)}{=} 0,$$

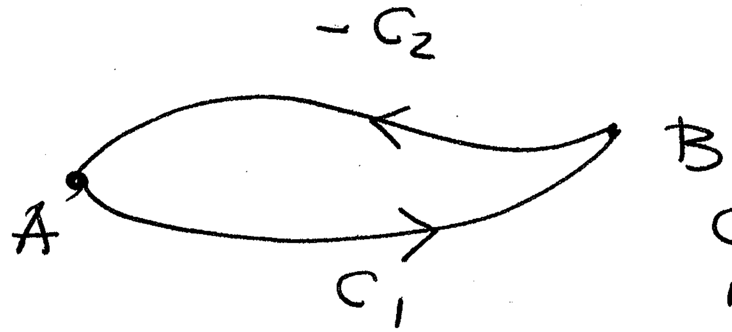
We proved that if $\int_C \vec{F}\cdot dr$ is path independent, then $\int_C \vec{F}\cdot d\vec{r} = 0$ whenever C is a closed curve.

We have to verify now that if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed C, then the integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent.

If C_1 and C_2 have the same endpoints



then the curve $C = C_1 + (-C_2)$



$C = C_1 + (-C_2)$
is closed

is closed (it starts at A and ends at A). Hence

$$0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

which proves path independence of the integral. $\square$

Recall that if $\vec{F} = \nabla f$ is a conservative vector field, then the integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

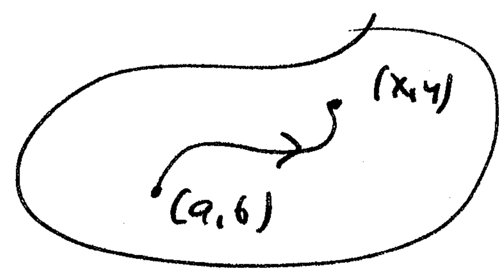
It turns out that path independency characterizes conservative vector fields.

Theorem Let $\vec{F}$ be a continuous vector field in a domain D. Then the integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent if and only if the vector field $\vec{F}$ is conservative i.e. if there is a function f (potential of $\vec{F}$) such that $\nabla f = \vec{F}$.

If $\int_C \vec{F}\cdot d\vec{r}$ is path independent we construct the potential f, $\nabla f = \vec{F}$ as follows.

We fix (a,b) in D and we define

$$(*) \quad f(x,y) := \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$$



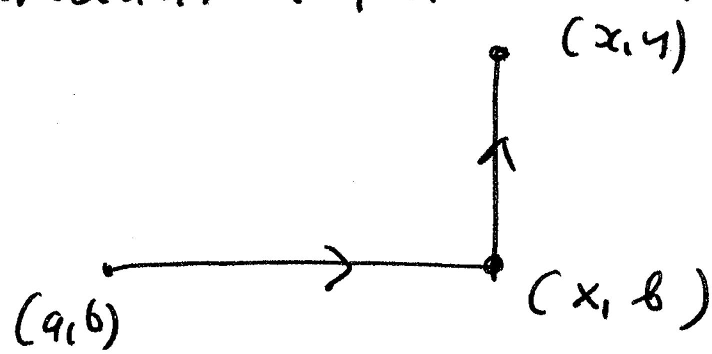
where the integral is along __any__ curve connecting (a,b) to (x,y). Clearly, the integral does not depend on which curve we choose, because it is path independent.

One can check that the function f defined by $(*)$ satisfies

$$\nabla f = \vec{F}$$

The formula $(*)$ is written in the case of 2 variables, but the argument applies to the case of 3 variables too.

Formula $(*)$ gives also a practical way of computing the potential f. We simply compute $f(x,y)$ by integrating $(*)$ along a curve on which the integral is easy to evaluate. For example



We will see later how to do it in practice.

Not every vector field is conservative or equivalently, not every integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

The next result shows a simple method of checking that a vector field is <u>not</u> conservative

<u>Theorem</u> If $\vec{F} = \langle P, Q \rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Indeed, if $\vec{F} = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ then $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$ and hence

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}. \quad \square$$

Therefore, if a vector field satisfies

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

~~at~~ somewhere, then $\vec{F}$ is <u>not</u> conservative so $\int_C \vec{F} \cdot d\vec{r}$ is <u>not</u> path indep.

Example Consider the vector field

$$\vec{F}(x,y) = \langle P, Q \rangle = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

This vector field is defined in

$$D = \{(x,y) : (x,y) \neq (0,0)\}$$

That is, it is defined everywhere except the origin. We have

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(-1)(x^2+y^2) - (-y)\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Therefore,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

However, the vector field is not conservative, because the integral along the closed curve

$$C: \quad \vec{r}(t) = \langle \cos t, \sin t \rangle, \ 0 \le t \le 2\pi$$

is not equal 0. Indeed,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy =$$

$$\int_0^{2\pi} -\frac{y(t)}{x^2(t)+y^2(t)}\, x'(t) + \frac{x(t)}{x^2(t)+y^2(t)}\, y'(t)\,dt =$$

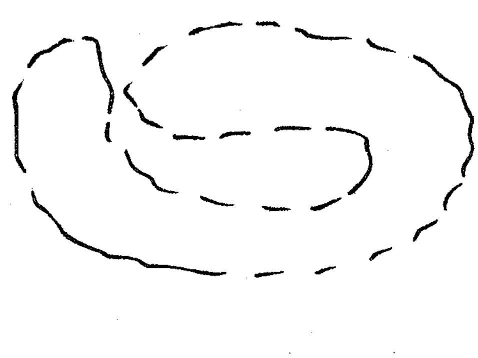
$$\underset{\substack{\uparrow \\ x^2(t)+y^2(t)=1 \\ \text{unit circle}}}{=} \int_0^{2\pi} -y(t)x'(t) + x(t)\,y'(t)\,dt =$$

$$= \int_0^{2\pi} \underbrace{-\sin t\,(-\sin t) + \cos t\cdot\cos t}_{\sin^2 t+\cos^2 t=1}\,dt = 2\pi.$$

$\square$

The domain in the above example has a hole – origin removed. Domains without holes are called simply connected. Here are examples

$\mathbb{R}^2$, $\quad \{(x,y) \mid x>0,\ y>0\}$



Domains with holes - not simply connected

$\{(x,y):\ 3 < x^2+y^2 < 10\}$

$\{(x,y):\ (x,y) \neq (0,0)\}$

or

While in general the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

does not guarantee that the vector field is conservative (see the example), it does guarantte it in the domain is simply connected - has no holes.

Theorem Assume that $\vec{F} = \langle P, Q \rangle$ is a vector field in a simply connected domain D. Then $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ in } D.$$

Now, we will show how to find the potential f, $\nabla f = \vec{F}$ in practice.

Example Determine whether or not the vector field

$$\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$$

is conservative.

Solution $P = x-y$, $Q = x-2$,

$P_y = -1$, $Q_x = 1$, $P_y \neq Q_x$

so the vector field is not conservative.

Example Show that the vector field

$$\vec{F}(x,y) = \langle 3+2xy,\ x^2-3y^2 \rangle$$

is conservative and find a potential of $\vec{F}$ i.e., f such that $\nabla f = \vec{F}$.

We will explain every step in the solution, but the solution in the next example will not contain that many details.

Solution $P = 3+2xy$, $Q = x^2-3y^2$

$$\frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x, \quad \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Since the vector field is defined in $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected, $\vec{F}$ is conservative. Now we will find f such that $\nabla f = \vec{F}$ i.e.

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle P, Q \rangle$$

i.e.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3+2xy \\ \dfrac{\partial f}{\partial y} = x^2-3y^2 \end{cases}$$

Consider the first equation

$$\frac{\partial f}{\partial x}(x,y) = 3 + 2xy.$$

Fix y. Then, it is an equation in one variable x and we can find f by integrating both sides with respect to x

$$f(x,y) = \int 3 + 2xy \, dx = 3x + x^2y + c.$$

However, for each y that we fix, we may obtain a different constant c so in fact c depends on y so it is not a constant, but a function of y so

$$f(x,y) = 3x + x^2y + c(y).$$

Now the second equation

$$\frac{\partial f}{\partial y}(x,y) = x^2 - 3y^2$$

yields

$$\frac{\partial}{\partial y}\left(3x + x^2y + c(y)\right) = x^2 - 3y^2$$

$$x^2 + c'(y) = x^2 - 3y^2$$

$$c'(y) = -3y^2$$
$$c(y) = -y^3 + d$$

constant

and we have

$$f(x,y) = 3x + x^2y - y^3 + d$$

Adding a constant d does not alter the value of the gradient so we can assume that $d=0$ (the potential is determined up to an additive constant - just like an antiderivative).

Thus

$$f(x,y) = 3x + x^2y - y^3$$

is a potential of $\vec{F}$.

We will check it to make sure we didn't make a mistake, but this step is not necessary

$$\nabla f = \langle f_x, f_y \rangle = \langle 3+2xy, x^2-3y^2 \rangle = \vec{F}$$

Similar method applies to vector fields in $\mathbb{R}^3$.

Example Find a function f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y,z) = \sin y\, \vec{i} + (x\cos y + \cos z)\, \vec{j} - y \sin z\, \vec{k}$$

and C is parametrized by

$$\vec{r}(t) = \sin t\, \vec{i} + t\, \vec{j} + 2t\, \vec{k}, \quad 0 \le t \le \frac{\pi}{2}$$

Solution We are solving equations

$$\begin{cases} f_x = \sin y \\ f_y = x\cos y + \cos z \\ f_z = -y \sin z \end{cases}$$

Integrating the first equation with respect to x yields

$$f = x \sin y + g(y,z)$$

g is a constant with respect to x but it may depend on y and z so it is a function of y and z

Now the second equation yields

$$f_y = x\cancel{\cos y} + g_y(y,z) = \cancel{x\cos y} + \cos z$$

$$g_y(y,z) = \cos z$$

Integrating with respect to y yields

$$g(y,z) = y\cos z + h(z)$$

Again, h is a constant with respect to y, but it may depend on z so it is a function of z:

Thus

$$f = x\sin y + y\cos z + h(z)$$

and the third equation gives

$$f_z = -y\sin z + h'(z) = -y\sin z$$

$$h'(z) = 0$$

$$h(z) = c \text{ - constant}$$

$$f(x,y,z) = x\sin y + y\cos z + c$$

and we can take $c = 0$

$$f(x,y,z) = x\sin y + y\cos z$$

Now we can evaluate the integral using the fundamental theorem of line integrals

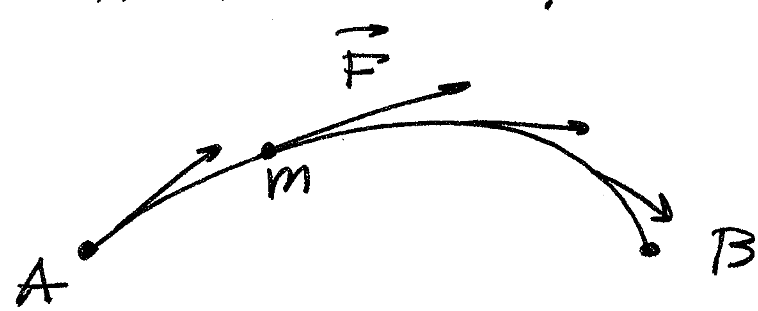
$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(\tfrac{\pi}{2})) - f(\vec{r}(0))$$

$$= f(1,\tfrac{\pi}{2},\pi) - f(0,0,0) =$$

$$1\cdot \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\pi - 0 = 1-\frac{\pi}{2}. \quad \square$$

Conservation of energy

Suppose that a force vector field $\vec{F}$ moves an object of mass m along a curve C: $\vec{r}(t)$, $a\le t\le b$ from $\vec{r}(a)=A$ to $\vec{r}(b)=B$.



According to the Second Law of Motion

$$\vec{F}(\vec{r}(t)) = \underbrace{m}_{\text{mass}}\ \underbrace{\vec{r}\,''(t)}_{\text{acceleration}}.$$

The work done by the force $\vec{F}$ on the object equals

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b m\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt = \heartsuit$$

$$\frac{d}{dt}\left(|\vec{r}\,'(t)|^2\right) = \frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) =$$

$$= \vec{r}\,''(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) =$$

$$= 2\,\vec{r}\,''(t) \cdot \vec{r}\,'(t).$$

Hence

$$r''(t) \cdot \vec{r}\,'(t) = \frac{1}{2}\frac{d}{dt}\left(|\vec{r}\,'(t)|^2\right).$$

Therefore,

$$\heartsuit = \frac{m}{2}\int_a^b \frac{d}{dt}\left(|\vec{r}\,'(t)|^2\right) \quad \underset{\text{Fundamental Theorem of Calculus}}{=}$$

$$= \frac{m}{2}\,|\vec{r}\,'(t)|^2\Big|_{t=a}^{t=b} = \frac{m}{2}\,|\underbrace{\vec{r}\,'(b)}_{\text{velocity at }B}|^2 - \frac{m}{2}\,|\underbrace{\vec{r}\,'(a)}_{\text{velocity at }A}|^2$$

Recall that the kinetic energy equals

$$K = \frac{m}{2}v^2,$$

hence

$$(*) \qquad W = \frac{m}{2}v(B)^2 - \frac{m}{2}v(A)^2 = K(B) - K(A)$$

We proved that the work done by $\vec{F}$ on the object equals to the increase of the kinetic energy.

Suppose now that the force field $\vec{F}$ is conservative i.e., $\vec{F} = \nabla f$ for some function f. In physics the potential energy of f equals $P = -f$ so $\vec{F} = -\nabla P$ and the Fundamental theorem of line integrals yields

$$W = \int_C \vec{F} \cdot d\vec{r} = -\int_C \nabla f \cdot d\vec{r} =$$

$$= -[P(B) - P(A)] = P(A) - P(B)$$

$$(**) \qquad W = P(A) - P(B)$$

Comparing (*) and (**) yields

$$K(B) - K(A) = P(A) - P(B)$$

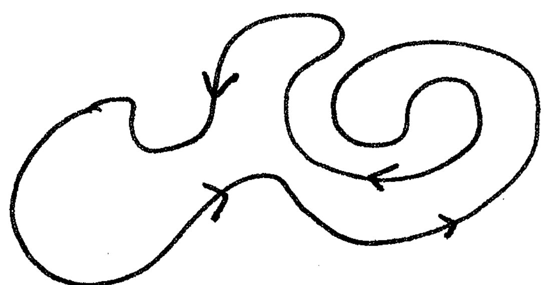
or

$$P(A) + K(A) = P(B) + K(B).$$

That is, in the presence of the conservative force field the total energy: potential + kinetic is preserved. This is the Law of Conservation of Energy. The name "conservative vector field" comes from this law: it is the vector field that preserves energy.

Green's Theorem

A simple closed curve is a closed curve without self-intersections

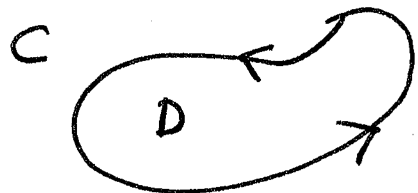


simple closed

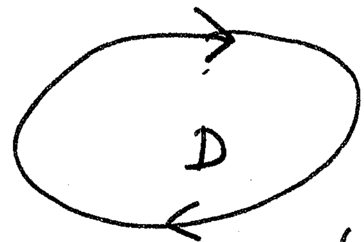
closed but not simple

If C is a simple closed curve, it bounds a region D that has no holes (so D is simply connected). Then C is the boundary of D.

We call the counterclockwise orientation of C <u>positive</u> and the clockwise orientation of C <u>negative</u>.



positive orientation of the bounday of D



negative orientation of the bounday of D.

In other words, we can describe the positive orientation of the boundary C of D as follows: as we walk along C, the domain D is on the left.

In what follows we will assume that the simple closed curve is smooth or piecewise-smooth.

Examples of positively oriented piecewise-smooth simple closed curves.

Theorem (Green's theorem)

Let C be positively oriented simple closed piecewise-smooth curve and assume that C bounds a region D. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

We assume here that the functions P and Q are defined in a region that contains D and that they have continuous partial derivatives.

Clearly, if C is negatively oriented, then

$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Example Evaluate

$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$$

where C is the circle $x^2+y^2=9$.

The circle $x^2+y^2=9$ is the boundary of the disc $D = \{(x,y) : x^2+y^2 \leq 9\}$ and we can have positive (counterclockwise) or negative (clockwise) orientation.

If a problem does not specify orientation of the curve, it is assumed that the orientation is positive

In the above example orientation of the circle is not mentioned so it is positive. Thus we could take $\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle, 0 \leq t \leq 2\pi$ and try to evaluate the resulting line integral. However, the expression we would have to evaluate would be vey complicated. On the other hand, as we shall see, Green's theorem provides a vey simple solution.

Solution Applying Green's theorem we have

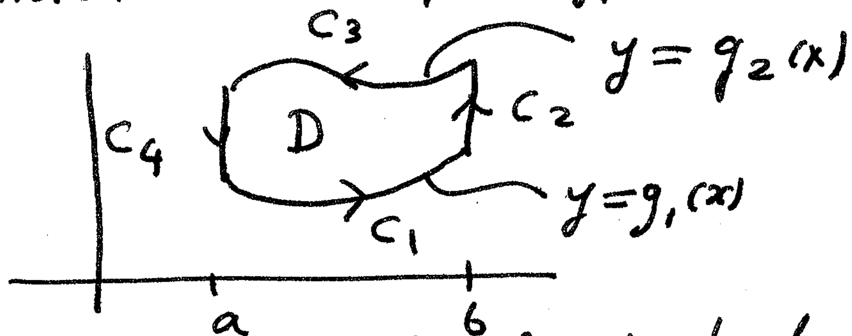
$$\int_C (\underbrace{3y - e^{\sin x}}_{P})\,dx + (\underbrace{7x + \sqrt{y^4+1}}_{Q})\,dy =$$

$$\iint_D \left(\frac{\partial}{\partial x}(\underbrace{7x + \sqrt{y^4+1}}_{Q}) - \frac{\partial}{\partial y}(\underbrace{3y - e^{\sin x}}_{P}) \right) dA =$$

$$\iint_D (7-3)\,dA = 4\,|D| = 4 \cdot \pi \cdot 3^2 = 36\pi.$$

Here $D = \{(x,y) : x^2 + y^2 \leq 9\}$ is the disc of radius 3 so its area equals $|D| = \pi \cdot 3^2$. □

Example Prove Green's theorem when the domain D is of type I:



and C is the positively oriented boundary of D

<u>Proof</u>. We have

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx =$$

$$\int_a^b P(x,y)\Big|_{y=g_1(x)}^{y=g_2(x)} dx =$$

$$\int_a^b \left(P(x, g_2(x)) - P(x, g_1(x)) \right) dx$$

$$(*) \quad \int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx.$$

$$\int_{C_1} P(x,y)\,dx = \heartsuit$$

$$\vec{r}(t) = \langle t, g_1(t) \rangle, \quad x'(t) = 1, \quad a \le t \le b$$

$$\heartsuit = \int_a^b P(t, g_1(t)) \cdot 1\,dt = \int_a^b P(x, g_1(x))\,dx.$$

Similarly

$$\int_{C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx.$$

The negative sine, because the orientation of C_3 is from right to left i.e., it is in the opposite direction to the direction of orientation of C_1.

The curves C_2, C_4 are parametrized by

$$\vec{r}(t) = \langle b, y(t)\rangle \quad \text{and} \quad \vec{r}(t) = \langle a, y(t)\rangle$$

respectively. While, we didn't write formulas for $y(t)$ explicitly, it is not needed, because in both cases $x'(t) = 0$ so

$$\int_{C_{2,4}} P\,dx = \int_\alpha^\beta P(x(t), y(t))\underbrace{x'(t)}_{0}\,dt = 0.$$

This should be clear geometrically. Along the vertical intervals C_2, C_4 there is no increase of x and the integrals $\int_C P\,dx$ are with the respect to the increases of the x coordinate.

Therefore $(*)$ yields

$$\int_C P\,dx = \int_a^b P(x, g_1(x))\,dx + 0 - \int_a^b P(x, g_2(x))\,dx + 0$$

$$= \int_a^b \big(P(x, g_1(x)) - P(x, g_2(x))\big)\,dx$$

$$= -\int_a^b \big(P(x, g_2(x)) - P(x, g_1(x))\big)\,dx,$$

so

$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA .$$

Similar calculation shows that

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

and hence

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \square$$

<u>Example</u> If C is as in Green's theorem and

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

then

$$\int_C P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA = 0 .$$

We could however, prove this without using Green's theorem. Indeed, D has no holes, it is simply connected and hence the vector field $\vec{F} = \langle P, Q \rangle$ is conservative (see p. 127)

Therefore,

$$\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{r} = 0,$$

because the integral of the conservative vector field is path independent (p. 116) so the integral along a closed curve equals 0 (p. 117).

$\square$

Green's theorem leads to a very important formula for the computation of the area of a domain.

<u>Theorem</u> Let C be a positively oriented simple closed piecewise-smooth curve C bounding a region D. Then

$$\text{Area}(D) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx.$$

<u>Proof</u> If $P = 0$, $Q = x$, then Green's theorem yields

$$\int_C x\,dy = \int_C P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{1} dA$$

$$= \iint_D dA = \text{Area}(D)$$

Similarly, if $P = -y$, $Q = 0$, then

$$\int_C -y\,dx = \int_C P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{1} dA = \text{Area}(D).$$

This proves the first two formulas in the theorem. Adding them up yields

$$2\,\text{Area}(D) = \int_C x\,dy - \int_C y\,dx$$

so

$$\text{Area}(D) = \frac{1}{2}\int_C x\,dy - y\,dx. \qquad \square$$

<u>Example</u> Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b > 0$$

<u>Solution.</u>

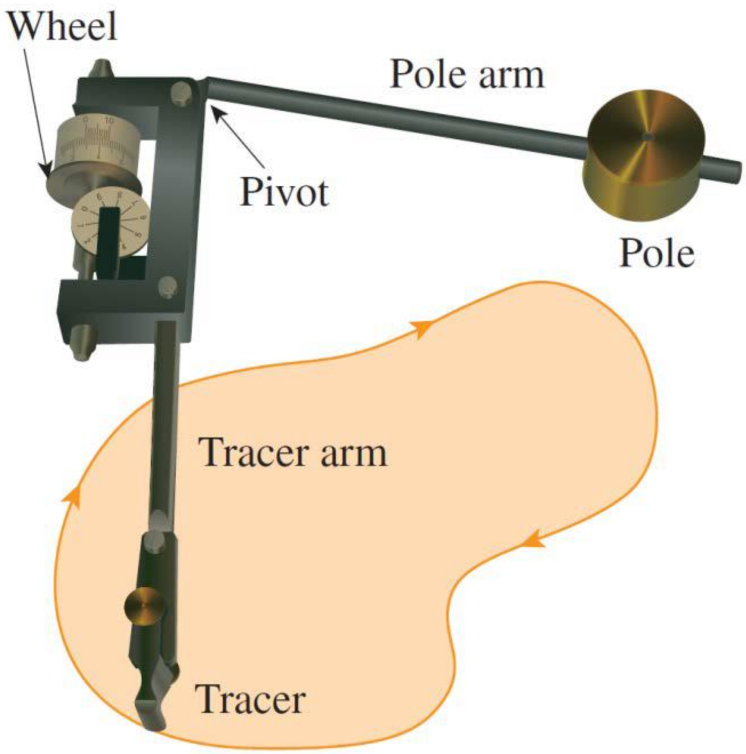
$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$, $0 \le t \le 2\pi$
is a positively oriented parametrization of the boundary of the ellipse. Hence

$$A = \frac{1}{2}\int_C x\,dy - y\,dx =$$

$$\frac{1}{2}\int_0^{2\pi} \underbrace{(a\cos t)}_{x(t)}\underbrace{(b\cos t)}_{y'(t)} - \underbrace{(b\sin t)}_{y(t)}\underbrace{(-a\sin t)}_{x'(t)}\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} \underbrace{\cos^2 t + \sin^2 t}_{1}\,dt = \boxed{\pi ab}. \quad \square$$

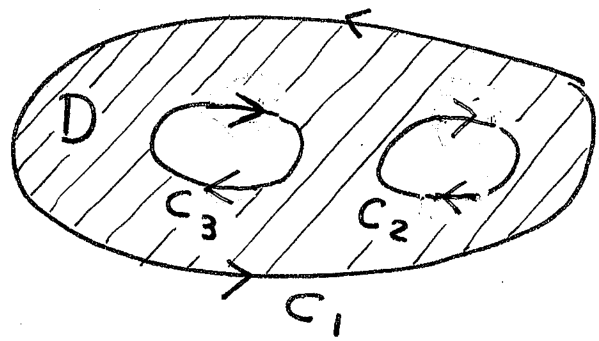
Example The above results show that integrating $\frac{1}{2}(x\,dy - y\,dx)$ along the boundary of D gives us the area of D. In fact there is a mechanical tool called planimeter that allows us to integrate $\frac{1}{2}\int_C x\,dy - y\,dx$ and thus to compute the enclosed area.



You simply move the tracer along the boundary and read the area from the counter (next to the wheel on the picture). The planimeters are useful in measuring the area of a region on a map. The first modern planimeter was built by the mathematician Jacob Amslet-Laffon in 1854.

Green's theorem in domains with holes.

Green's theorem tells us how to turn the line integral along the boundary into a double integral over the domain. However, it applies to the situations when the boundary of the domain is a simple curve and the domain has no holes. Consider now the domain



The boundary of D consists of three curves $C = C_1 + C_2 + C_3$. Note that the curves C_2 and C_3 are oriented clockwise. This is because the rule about the positive orientation of the boundary is as follows:

As we walk along the boundary, the domain is on the left

That means, the outer component of the boundary is oriented counterclockwise but the inner components are oriented clockwise. Thus the boundary $C = C_1 + C_2 + C_3$ of D as shown on the picture is positively oriented.

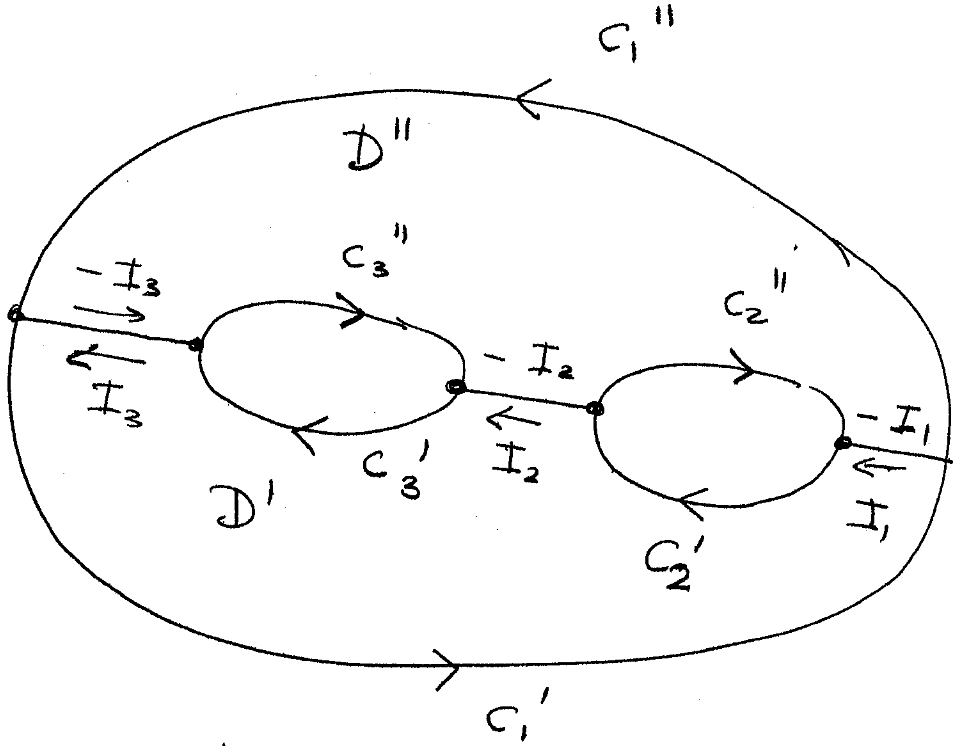
In this situation Green's theorem takes the following form

$$\boxed{\int_C P\,dx + Q\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Similar formula applies to domains with a larger number of holes.

Proof A beautiful trick allows us to split the domain D into two domains $D = D' + D''$ so that the boundaries of the domains D' and D'' are simple curves and we can apply Green's theorem to D' and D''.

We represent the domains D' and D'' on the next picture.



Boundary of $D' = C_1' + I_1 + C_2' + I_2 + C_3' + I_3$

Boundary of $D'' = C_1'' + (-I_3) + C_3'' + (-I_2) + C_2'' + (-I_1)$

The boundaries of the domains D' and D'' are simple curves so Green's theorem yields

$$\iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1'} + \int_{I_1} + \int_{C_2'} + \int_{I_2} + \int_{C_3'} + \int_{I_3} P\,dx + Q\,dy$$

$$\iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1''} - \int_{I_3} + \int_{C_3''} - \int_{I_2} + \int_{C_2''} - \int_{I_1} P\,dx + Q\,dy.$$

We add the two equalities and note that the integrals over the I intervals will cancel out

$$\int_{I_1} + \int_{I_2} + \int_{I_3} - \int_{I_3} - \int_{I_2} - \int_{I_1} P\,dx + Q\,dy = 0$$

Therefore,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} + \iint_{D''} =$$

$$\int_{C_1'} + \int_{C_2'} + \int_{C_3'} + \int_{C_1''} + \int_{C_2''} + \int_{C_3''} =$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy,$$

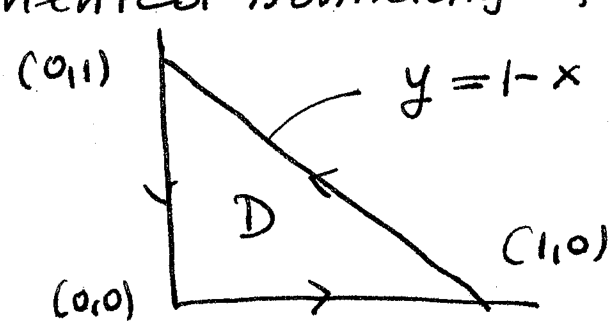
because $C_1 = C_1' + C_1''$, $C_2 = C_2' + C_2''$, $C_3 = C_3' + C_3''$.

$\square$

Three more examples

Example Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of three line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

Solution The curve C is the positively oriented boundary of the triangle



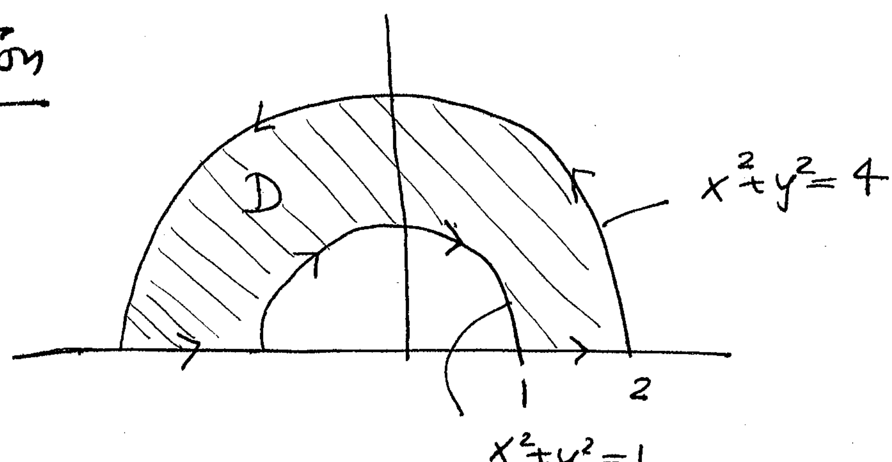
Thus Green's theorem yields

$$\int_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy = \iint_D \frac{\partial}{\partial x}\underbrace{(xy)}_{Q} - \frac{\partial}{\partial y}\underbrace{(x^4)}_{P}\,dA$$

$$= \iint_D y\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{6}. \quad \square$$

<u>Example</u> Evaluate $\int_C y^2dx + 3xy\,dy$, where C is the boundary of the semi-annular region D in the upper-half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

<u>Solution</u>



Since orientation of the boundary is not mentioned, it is implied that the orientation is positive.

The region D can be easily described in polar coordinates

$$D = \{(r,\theta) \mid 0 \le \theta \le \pi,\ 1 \le r \le 2\}$$

Therefore, Green's theorem yields

$$\int_C y^2 dx + 3xy\, dy = \iint_D \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\, dA$$

$$= \iint_D y\, dA = \int_0^{\pi}\int_1^2 \underbrace{r\sin\theta}_{y} \cdot \underbrace{r\, dr\, d\theta}_{dA}$$

$$= \int_0^{\pi} \sin\theta\, d\theta \int_1^2 r^2 dr = \frac{14}{3}. \qquad \square$$

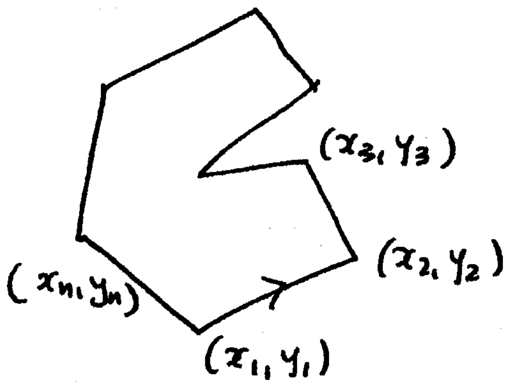
<u>Theorem</u> If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, then the area of the polygon equals

$$A = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation

$$x_{n+1} = x_1, \quad y_{n+1} = y_1.$$

Proof



$$A = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \sum_{i=1}^{n} \int_{(x_i, y_i)}^{(x_{i+1}, y_{i+1})} x\,dy - y\,dx$$

$\alpha(t) = (x_i + t(x_{i+1} - x_i),\ y_i + t(y_{i+1} - y_i)),\ t \in [0,1]$
is a parametrization of the segment connecting (x_i, y_i) to (x_{i+1}, y_{i+1}).

We have

$$\int_{(x_i, y_i)}^{(x_{i+1}, y_{i+1})} x\,dy - y\,dx = \int_0^1 (x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i) - (y_i + t(y_{i+1} - y_i))(x_{i+1} - x_i)\,dt$$

$$= \left(t x_i + \frac{t^2}{2}(x_{i+1} - x_i)\right)(y_{i+1} - y_i) - \left(t y_i + \frac{t^2}{2}(y_{i+1} - y_i)\right)(x_{i+1} - x_i)\Big|_0^1$$

$$= \left(x_i + \frac{1}{2}(x_{i+1} - x_i)\right)(y_{i+1} - y_i) - \left(y_i + \frac{1}{2}(y_{i+1} - y_i)\right)(x_{i+1} - x_i)$$

$$= \frac{1}{2}(x_i + x_{i+1})(y_{i+1} - y_i) - \frac{1}{2}(y_i + y_{i+1})(x_{i+1} - x_i)$$

$$= \frac{1}{2}\left(x_i y_{i+1} - \cancel{x_i y_i} + \cancel{x_{i+1} y_{i+1}} - x_{i+1} y_i - y_i x_{i+1} + \cancel{y_i x_i} - \cancel{y_{i+1} x_{i+1}} + y_{i+1} x_i\right)$$

$$= x_i y_{i+1} - x_{i+1} y_i .$$

Hence

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i).$$